Completing the Square

This guide describes the algebraic technique of completing the square and shows how to use it to solve quadratic equations.

Introduction

Completing the square is a more advanced algebraic technique which is extremely useful in solving **quadratic equations** and also plays a part in strategies for solving more complex mathematical problems. It is also used to derive the quadratic formula (see study guide: *Solving Quadratic Equations using the Quadratic Formula*). A good understanding of this study guide relies on many algebraic techniques described in other *Steps into Algebra* study guides. If you read through this sheet and understand a majority of the manipulations and arguments involved then you may consider yourself to be proficient at algebra. If you find sections difficult to understand, return to those areas and read the relevant study guide.

The method of completing the square

The method of completing the square is best explained by using an example.

Example: Solve $3x^2 + 12x - 5 = 0$.

Every quadratic equation contains either the mathematics $(x+n)^2$ or $(x-n)^2$ concealed within it. The bracketed expressions are the *squares* of completing the square and the value of *n* is determined by the particular quadratic equation you are investigating.

The first stage of completing the square is to identify $(x+n)^2$ or $(x-n)^2$ within the quadratic equation you are solving. The task of finding either $(x+n)^2$ or $(x-n)^2$ is made simpler by rearranging your quadratic equation to the form $x^2 + bx + c = 0$, if it is not in this form already.

The example is: solve $3x^2 + 12x - 5 = 0$ which is not in the form $x^2 + bx + c = 0$ as the coefficient of x^2 is 3 and not 1 as required. Part of the first stage of completing the square is to **divide the quadratic equation by the coefficient of x^2 unless the coefficient is 1**. In this case the coefficient is 3 and so, after dividing $3x^2 + 12x - 5 = 0$ by 3, you get:

$$x^2 + 4x - \tfrac{5}{3} = 0$$

which is now of the form $x^2 + bx + c = 0$ with $b = +4$ and $c = -5/3$.

Returning to $(x+n)^2$ and $(x-n)^2$, if you expand the brackets you get:

$$(x+n)^2 = x^2 + 2nx + n^2 \qquad \text{and} \qquad (x-n)^2 = x^2 - 2nx + n^2$$

which are both quadratic expressions of the form $x^2 + bx + c$ with $b = +2n$ in the first expansion, and $b = -2n$ in the second expansion.

The value of *b* in the quadratic equation you are solving determines which of the two squares to use. If *b* is positive you use $(x+n)^2$ and *b* is negative you use $(x-n)^2$. In the example $b = +4$ and so you choose $(x+n)^2$. You now need to calculate *n*. As $b = +2n = +4$ it follows that $n = +2$. In fact the value of *n* is always half of *b*. You can use the value of *n* to find that the square in *this* specific question is $(x+2)^2$. However, be careful; you are not saying that $x^2 + 4x - \tfrac{5}{3} = (x+2)^2$ merely that $x^2 + 4x - \tfrac{5}{3}$ *contains* $(x+2)^2$.

The next step is to **adjust your mathematics to account for the difference between the quadratic equation you are solving and the square you have identified**. In this example you have the square:

$$(x+2)^2 = x^2 + 4x + 4$$

But the expression in question is: $\qquad x^2 + 4x - \tfrac{5}{3}$

As you can see the first two terms (x^2 and $4x$) are correct but the constant $-5/3$ is incorrect. To correct this, add and subtract the constant generated by expanding the square (in this case 4) to the expression in question. You can then change the order of the terms to identify the square (bracketed in mathematics below):

$$x^2 + 4x - \tfrac{5}{3} = x^2 + 4x - \tfrac{5}{3} + 4 - 4 = \left(x^2 + 4x + 4\right) - 4 - \tfrac{5}{3} = (x+2)^2 - \tfrac{17}{3}$$

In other words you have written the quadratic expression as the square and an extra constant term. This is what completing the square is: you have identified a squared term in a quadratic and *completed* it by adjusting the constant.

But why should you do this? In its original form a quadratic equation cannot be solved by algebraic manipulation, but after completing the square it can. Returning to the example, after completing the square you find that:

$$(x+2)^2 - \tfrac{17}{3} = 0$$
$$(x+2)^2 = \tfrac{17}{3}$$
$$x + 2 = \pm\sqrt{\tfrac{17}{3}}$$
$$x = \pm\sqrt{\tfrac{17}{3}} - 2$$

Completing the square

1. Divide the quadratic equation by the coefficient of x^2.
2. Use to coefficient of x to choose the correct square either $(x+n)^2$ or $(x-n)^2$.
3. Halve the value of the coefficient of x to find n.
4. Expand the square to find the constant.
5. Adjust the constant to match the quadratic expression in question.
6. Solve for x.

Derivation of the quadratic formula

The famous quadratic formula for solving quadratic equations is derived by applying the **completing the square** method to the general quadratic equation $ax^2 + bx + c = 0$.

Step 1. Divide by a to give $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$.

Step 2. Choose $(x+n)^2$.

Step 3. Halve the coefficient of x. Here the coefficient of x is $\frac{b}{a}$ and so $n = \frac{b}{2a}$.

Step 4. The square is $\left(x + \frac{b}{2a}\right)^2$ which equals $x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}$ after expansion.

Step 5. Adding and subtracting the constant $\frac{b^2}{4a^2}$ to and from $x^2 + \frac{b}{a}x + \frac{c}{a}$ gives the appropriate balance:

$$x^2+\frac{b}{a}x+\frac{c}{a}+\frac{b^2}{4a^2}-\frac{b^2}{4a^2}=\left(x^2+\frac{b}{a}x+\frac{b^2}{4a^2}\right)+\frac{c}{a}-\frac{b^2}{4a^2}$$

$$=\left(x+\frac{b}{2a}\right)^2+\frac{c}{a}-\frac{b^2}{4a^2}$$

$$=\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}$$

Step 6. Solve for x:

$\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}=0$	
$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$	Subtract $\frac{4ac-b^2}{4a^2}$
$x+\frac{b}{2a}=\frac{\pm\sqrt{b^2-4ac}}{2a}$	Take the square root
$x=\frac{\pm\sqrt{b^2-4ac}}{2a}-\frac{b}{2a}$	Subtract $\frac{b}{2a}$
$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$	Combine algebraic fractions

which results in the well-known quadratic formula.

Want to know more?

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- Call: 01603 592761
- Ask: ask.let@uea.ac.uk
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