# Learning Enhancement Team

University of East Anglia
STUDENT SUPPORT SERVICE

***Steps into Algebra***

# Laws of Indices

***This guide describes how to work with and manipulate the laws of indices in mathematics.***

## What are indices?

**Indices** usually appear in mathematics as a numerical superscript after a number or variable and are also known as **powers** or **exponents**. The word 'indices' is the plural of 'index'. The most common use of an index is to represent multiplication of a variable or number by itself a number of times. They can also be used to describe reciprocals and roots. Some examples are $10^3$, $x^{-1}$ and $p^{1/2}$. The index can be any positive or negative number or zero. Different types of numerical indices represent different types of mathematics and this guide looks at each of them in detail. A number or variable with an index is called the **base** so 10 is the base in $10^3$ and $x$ is the base in $x^{-1}$.

## Positive whole number indices

Positive *whole* number indices represent the number of times a variable or number is multiplied by itself. It is important not to confuse this with addition as this leads to very common errors. (From now on the notation '$\cdot$' will be used for multiplication instead of '$\times$' to stop any confusion between multiplication and the variable $x$.)

*Example*: What is $x \cdot x \cdot x \cdot x$ expressed in index form?

The variable $x$ is multiplied by itself four times and so the index form is $x^4$. You can say '$x$ to the power four' or simply '$x$ to the four'; try not to say '$x$ four' as this can lead to confusion.

You can see why index notation is beneficial as it saves time and space in writing mathematics (think about what $x^{25}$ represents).

Do not confuse $x^4$ with four lots of $x$ as this is $x + x + x + x = 4x$, something completely different.

*Example*: Calculate $3^2$ and $5^3$.

$3^2 = 3 \cdot 3$ which is 9. An index of 2 has a special name and $3^2$ is said 'three squared'. You can think of $3^2$ as the area of a square with side length of 3.

$5^3 = 5 \cdot 5 \cdot 5 = 125$. The index of 3 also has a special name and $5^3$ is said 'five cubed'. $5^3$ can thought of as the volume of a cube with side length of 5.

If you are comfortable with the above examples you can deduce the first law of indices.

*Example*: What is $x^2 \cdot x^3$ written as a single index?

If you write out $x^2$ and $x^3$ in full you find that:

$$x^2 \cdot x^3 = (x \cdot x) \cdot (x \cdot x \cdot x)$$

The brackets have been introduced to help you see what has happened but really there is no need for them as multiplication can be carried out in any order. So really:

$$x^2 \cdot x^3 = x \cdot x \cdot x \cdot x \cdot x = x^5$$

Similarly:

$$x^2 \cdot x^2 = x \cdot x \cdot x \cdot x = x^4 \quad \text{and} \quad x^2 \cdot x^4 = x \cdot x \cdot x \cdot x \cdot x \cdot x = x^6$$

Can you see a pattern emerging? To get the index of the answer you add the indices in the question. Mathematically this can be written as:

$$\boxed{x^m x^n = x^{m+n}}$$

which is the **first law of indices**. Importantly $x$ is the base throughout and this must be the case for you to use the first law of indices. So, for example, $p^3 p^8 = p^{3+8} = p^{11}$ however $p^3 q^8$ cannot be simplified using the law as we have different bases ($p$ and $q$). The first law of indices is true when $m$ and $n$ are *any number* either positive, negative or a fraction.

You can deduce an important fact about indices from the first law. Think about each of the following: $x \cdot x^2$ and $x \cdot x^4$. By writing out the mathematics in full can you see that $x \cdot x^2 = x^3$ and $x \cdot x^4 = x^5$? The only way that the first law of indices remains true is if that:

$$\boxed{x = x^1}$$

so $x \cdot x^2 = x^1 \cdot x^2 = x^{1+2} = x^3$ and $x \cdot x^4 = x^1 \cdot x^4 = x^{1+4} = x^5$. It is true that $x = x^1$ which can be thought of as 'anything to the power 1 is itself'. This is really useful when manipulating indices.

## Negative whole number indices

You have seen how the first law of indices can be deduced from multiplying variables together. Another law can be found from the division of variables. (If you find the following difficult read the study guide: *Cancelling Down Fractions*.)

*Example*: Simplify $x^5 \div x^3$.

In algebra you will rarely see the symbol '÷' as division is almost always implied by writing an expression as a fraction. So, doing this and cancelling down $x^5 \div x^3$ is:

$$\frac{x \cdot x \cdot \not{x} \cdot \not{x} \cdot \not{x}}{\not{x} \cdot \not{x} \cdot \not{x}} = \frac{x \cdot x}{1} = x^2 \qquad \text{or} \qquad \frac{x^5}{x^3} = x^2$$

Again, can you see a pattern? So, **when multiplying variables you add the indices** and **when dividing variables you subtract the indices**. So, in general:

$$\boxed{\frac{x^m}{x^n} = x^{m-n}}$$

which is the **second law of indices**. Again the bases *must be the same* for this to work. You can deduce two useful facts from this law. Let's think about $x^3 \div x^3$. Written out in full you find that:

$$\frac{\not{x} \cdot \not{x} \cdot \not{x}}{\not{x} \cdot \not{x} \cdot \not{x}} = \frac{1}{1} = 1 \qquad \text{and using the law} \qquad \frac{x^3}{x^3} = x^{3-3} = x^0$$

which implies that:

$$\boxed{x^0 = 1}$$

This is true for any $x$ (except 0) in order for the laws of indices to hold. This equation says that 'anything to the power 0 is equal to 1'. This is extremely useful in algebra.

What happens if the index associated with the denominator is larger than that associated with the numerator? Think about $x^3 \div x^5$. Writing out in full you find that:

$$\frac{\not{x}\cdot\not{x}\cdot\not{x}}{x\cdot x\cdot\not{x}\cdot\not{x}\cdot\not{x}}=\frac{1}{x\cdot x}=\frac{1}{x^2} \qquad \text{or, using the second law} \qquad \frac{x^3}{x^5}=x^{-2}$$

This example implies that $\frac{1}{x^2}=x^{-2}$. In general:

$$\boxed{\frac{1}{x^n}=x^{-n}}$$

where $x$ can be any number except for 0 and allows you to express **reciprocals** in terms of indices, in other words reciprocals are represented by indices of opposite sign.

*Example*: Calculate $\frac{4^3}{4^5}$.

Using the second law, $\frac{4^3}{4^5}=4^{3-5}=4^{-2}$.

You can use the reciprocal law with $n=2$ to calculate that $4^{-2}=\frac{1}{4^2}=\frac{1}{16}$.

You can also use the reciprocal law to show that the first and second laws of indices are connected to each other; in fact the second law is a reinterpretation of the first. Start by writing the second law as a product of two fractions:

$$\frac{x^m}{x^n}=\frac{x^m}{1}\cdot\frac{1}{x^n}=x^m\cdot\frac{1}{x^n} \quad ,$$

now re-writing the reciprocal $\frac{1}{x^n}$ as $x^{-n}$ and then using the first law gives you:

$$x^m\cdot\frac{1}{x^n}=x^m\cdot x^{-n}=x^{m+(-n)}=x^{m-n} \quad ,$$

which is the second law.

## Indices raised to another index

The laws of indices can be used to simplify the square of a square or a cube of a square and so on.

*Example*: Simplify $\left(x^3\right)^4$.

The question asks the result of raising $x^3$ to the power 4. You can write this in full as:

$$\left(x^3\right)^4=\left(x^3\right)\cdot\left(x^3\right)\cdot\left(x^3\right)\cdot\left(x^3\right)=(x\cdot x\cdot x)\cdot(x\cdot x\cdot x)\cdot(x\cdot x\cdot x)\cdot(x\cdot x\cdot x)=x^{12}$$

which is long and tedious and you could make a mistake. However, there is a pattern; multiplication of the indices in the original question also results in 12. This leads to another law of indices. In general:

$$\left(x^m\right)^n = x^{m\cdot n}$$

which is the **third law of indices**. You *can* apply this law to more general cases which have *different* bases. In general:

$$\left(x^m y^n\right)^p = x^{m\cdot p} y^{n\cdot p}$$

and similarly

$$\left(\frac{x^m}{y^n}\right)^p = \frac{x^{m\cdot p}}{y^{n\cdot p}}$$

To use these laws identify *m*, *n* and *p* on the left-hand side of the equals sign and use them to calculate the relevant indices on the right-hand side.

*Example*: Simplify $\left(s^2 t^{-3}\right)^5$.

Using the first rule $\left(x^m y^n\right)^p = x^{m\cdot p} y^{n\cdot p}$ with $m = 2$, $n = -3$ and $p = 5$ gives:

$$\left(s^2 t^{-3}\right)^5 = s^{2\cdot 5} t^{-3\cdot 5} = s^{10} t^{-15} = \frac{s^{10}}{t^{15}}$$

## Fractional indices

You have seen that integer indices represent different ways of manipulating numbers and variables. This section looks at fractional indices. Let's begin with a special type of fractional index, where the numerator is equal to one. You have already seen that a variable (or number) multiplied by itself is written as $x^2$ and can be thought of as representing a square with sides of length *x*. In other words the length *x* is the **root** from which the square is built. For example, take the number 9. $9 = 3 \cdot 3 = 3^2$. A square with area 9 has a side of length 3; 3 is said to be the **square root** of 9. This is given a special symbol $\sqrt[2]{\ }$ or simply $\sqrt{\ }$. [Note that $-3$ is also a square root of 9 as $-3 \cdot -3 = 9$, it useful here to restrict our discussion to positive numbers only.] Similarly 3 is the **cube root** of 27 as a cube of volume 27 has the side length of 3. Cube root has the symbol $\sqrt[3]{\ }$. The higher roots, (fourth, fifth and so on) represent shapes outside of 3 dimensions but the principle is the same.

It is possible to express these roots as indices using the first law. Let's return to the number 9; knowing that $9 = 3 \cdot 3$ and letting $3 = 9^m$ (where *m* is unknown) you can write:

$$9 = 3 \cdot 3 = 9^m \cdot 9^m$$

As $9 = 9^1$ the equation above implies that $9^1 = 9^m \cdot 9^m$. From the first law of indices $m + m = 1$ and so $2m = 1$ or $m = \frac{1}{2}$. Therefore the square root is denoted by the index of a half.

You can follow this procedure for any root and determine that, for the $m^{\text{th}}$ root;

$$\sqrt[m]{x} = x^{1/m}$$

where the numerator in the index is always 1.

Finally let's look at fractional indices where the numerator is not 1. To understand how to manipulate these indices it is useful to remember that the fraction $m/n$ can always be written as $m \cdot \frac{1}{n}$. Remember that the third law of indices has the indices multiplied together. Using these two facts allows a final general rule to be written as:

$$x^{m/n} = \left(x^{1/n}\right)^m \qquad \text{or equivalently} \qquad x^{m/n} = \left(x^m\right)^{1/n}.$$

## Want to know more?

If you have any further questions about this topic you can make an appointment to see a **Learning Enhancement Tutor** in the **Student Support Service**, as well as speaking to your lecturer or adviser.

- Call: 01603 592761
- Ask: ask.let@uea.ac.uk
- Click: https://portal.uea.ac.uk/student-support-service/learning-enhancement

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