Learning Enhancement Team

University of East Anglia
STUDENT SUPPORT SERVICE

Steps into Differential Equations

Homogeneous Second Order Differential Equations

This guide helps you to identify and solve homogeneous second order ordinary differential equations.

Introduction

A **differential equation** (or **DE**) is any equation which contains derivatives, see study guide: *Basics of Differential Equations*. To make the best use of this guide you will need to be familiar with some of the terms used to categorise differential equations.

Linear DE: The function *y* and any of its derivatives can **only** be multiplied by a constant or a function of *x*.

Ordinary differential equation (ODE): Contains only ordinary derivatives.

Second order DE: Highest derivative is second order: $\frac{d^2y}{dx^2}$ or y' or $\ddot{x}$

Top Tip: You may see the term **homogeneous** used to describe first order differential equations. For more information, see study guide: *Homogeneous First Order Differential Equations.*

However, there is an entirely different meaning for a **homogeneous** second order differential equation. This guide is only concerned with linear second order ODEs and the examples that follow will concern a variable *y* which is itself a function of a variable *x*.

A linear second order ordinary differential equation is called **homogeneous** if the ODE contains only terms proportional to y and derivatives of y, and there are no terms that are a function of *x* alone. That is to say, if the DE is homogeneous then it can be written in the form:

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0$$

where a and b can be constants, or functions of x.

Examples:

$$\frac{d^2y}{dx^2}+3\frac{dy}{dx}+y=0$$

Homogeneous second order ODE – only contains terms proportional to y and its derivatives y' and y''.

$$\frac{d^2y}{dx^2}+\frac{dy}{dx}+y=2x$$

Inhomogeneous (not homogeneous) second order ODE – contains an independent function of x, (the $2x$ on the right-hand side of the equation).

$$\frac{d^2y}{dx^2}+\frac{dy}{dx}+3xy=0$$

Homogeneous second order ODE – contains only terms proportional to y, y' and y'', the function of x is a coefficient of the function of y, and not an independent function.

A constant term can also result in an ODE being inhomogeneous. The ordinary differential equation:

$$\frac{d^2y}{dx^2}+\frac{dy}{dx}+y=8$$

is **not** homogeneous because of the constant term on the right-hand side.

Example: Which of these second order ordinary differential equations are homogeneous?

(a) $\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}=2xy$ (b) $\dfrac{d^2y}{dx^2}+3\dfrac{dy}{dx}+2x=0$

Solutions:

(a) $$\frac{d^2y}{dx^2}+\frac{dy}{dx}=2xy$$

You can rearrange the ODE to look like: $\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}-2xy=0$ which is now in the form $\dfrac{d^2y}{dx^2}+a\dfrac{dy}{dx}+by=0$ where $a=1$ and $b=-2x$. There are no independent functions of x, and therefore it **is** homogeneous.

(b) $$\frac{d^2y}{dx^2}+3\frac{dy}{dx}+2x=0$$

You cannot write this ODE in the form: $\dfrac{d^2y}{dx^2}+a\dfrac{dy}{dx}+by=0$ as it contains $2x$ which is an independent function of *x.* Therefore, this ODE is **not** homogeneous.

Solving homogeneous second order ODEs

Due to the nature of differential equations, there may be more than one method to solve second order ODEs depending on their complexity. This guide will give a method for solving homogeneous linear second order ODEs with **constant coefficients only** (*a* and *b* are constants). Since these are second order differential equations, they will **always have two solutions**.

Step 1: Check your ODE is homogeneous.

Step 2: Write $y = e^{kx}$ where *k* is an unknown constant to be determined.

Step 3: Calculate derivatives of $y = e^{kx}$ to get $y' = ke^{kx}$ and $y'' = k^2e^{kx}$, and substitute these into your differential equation.

Step 4: As $e^{kx} \neq 0$ and is common to every term, you can divide through by e^{kx}.

Step 5: Solve the remaining **quadratic equation** for *k*. You will get two solutions: k_1 and k_2. (For more information on solving quadratic equations see study guides: *Solving Quadratic Equations by Factorisation*, *Solving Quadratic Equations using the Quadratic Formula*.)

Step 6: There are now **three** cases depending on the two solutions to the quadratic equation:

Case 1: When k has **two distinct real roots** $(k_1 \neq k_2)$.

Substitute your solutions for k into $y = e^{kx}$ to get $y_1 = e^{k_1 x}$, $y_2 = e^{k_2 x}$. These are two solutions to your ODE.

You can multiply either of these solutions by a constant and *still* get a solution to the ODE. Both of these solutions give zero when substituted into the ODE and so you can add them together and *still* get a solution (as zero plus zero is zero). This is called the **general solution**, and looks like:

$$y = Ae^{k_1 x} + Be^{k_2 x}$$

Case 2: When k has **one repeated root** $(k_1 = k_2)$

You should have one solution for *k* so substitute this into $y = e^{kx}$. So $y_1 = e^{k_1 x}$

is one solution to the ODE, however, second order ODEs **always** have two solutions. When you have one solution for k, the second solution to the ODE is obtained by multiplying $y_1 = e^{kx}$ by x to get $y_2 = xe^{kx}$. As with Case 1, you can multiply both solutions by a constant and add them together and still get a solution. So the general solution for Case 2 is:

$$y = Ae^{kx} + Bxe^{kx} = (A + Bx)e^{kx}$$

Case 3: When k has **complex roots**, ($k_1 = a + bi$ and $k_2 = a - bi$)

You have two complex solutions for k where a is the real part and $\pm b$ is the imaginary part. You can substitute $k = a \pm bi$ into the general solution for Case 1, and then use **Euler's Identity** to rewrite your solutions in terms of sin and cos (see study guide: *Euler's Formula and De Moivre's Theorem*). When you do this your general solution will for Case 3 will be:

$$y = Ae^{ax}\cos(bx) + Be^{ax}\sin(bx)$$

In practice, you can substitute a and b straight into this general solution.

Example: Solve $\dfrac{d^2y}{dx^2} - 9\dfrac{dy}{dx} + 18y = 0$.

Step 1: Here $a = -9$ and $b = 18$. There are no independent functions of x so the ODE is homogeneous.

Step 2: Next write $y = e^{kx}$ where k is an unknown constant to be determined.

Step 3: Calculate derivatives of $y = e^{kx}$: $y' = ke^{kx}$ and $y'' = k^2e^{kx}$. Substitute these into your differential equation to get:

$$k^2e^{kx} - 9ke^{kx} + 18e^{kx} = 0$$

Step 4: You can notice here that e^{kx} is common to each term. Since $e^{kx} \neq 0$ you can divide through by e^{kx} to get:

$k^2 - 9k + 18 = 0$, and you are left with a **quadratic equation** for k.

Step 5: You can then factorise the quadratic equation:

$k^2 - 9k + 18 = (k - 3)(k - 6) = 0$ so $k = 3$ or $k = 6$.

Step 6: You have **two distinct real roots**, so substitute these into $y = e^{kx}$ to get $y_1 = e^{3x}$ and $y_2 = e^{6x}$. These are both **solutions** to the ODE.

You can check this by substituting them into the original ODE:

Try $y_1 = e^{3x}$. Then $y' = 3e^{3x}$ and $y'' = 9e^{3x}$. Substituting into the original ODE $y'' - 9y' + 18y = 0$ gives: $9e^{3x} - 27e^{3x} + 18e^{3x} = 0$, which is true.

Next try $y_2 = e^{6x}$. Then $y' = 6e^{6x}$ and $y'' = 36e^{6x}$. Substituting into the original ODE gives: $36e^{6x} = 54e^{6x} - 18e^{6x}$ which is also true.

Since these are both solutions to the ODE you can multiply each solution by a constant and add them together to get:

$y = Ae^{3x} + Be^{6x}$ the **general solution** to $\dfrac{d^2y}{dx^2} - 9\dfrac{dy}{dx} + 18y = 0$

Solving homogeneous second order ODEs: Particular solutions (using initial conditions or boundary conditions)

If more is known about the situation that a differential equation describes, you may have extra information available which enables you to calculate the constants in your general solution. These conditions are called **initial conditions** or **boundary conditions** and are given as mathematical statements relating to the DE you are solving.

Example: Solve $\dfrac{d^2y}{dx^2} = 9\dfrac{dy}{dx} - 18y$ subject to initial conditions $y(0) = 3$ and $y'(0) = 15$

$y(0) = 3$	tells you that when	$x = 0$, $y = 3$
$y'(0) = 15$	tells you that when	$x = 0$, $y' = 15$

This ODE has already been solved on pages 4 and 5 of this guide. The **general solution** to this ODE is $y = Ae^{3x} + Be^{6x}$. However, you can now use the initial conditions to obtain values for the unknown constants *A* and *B*.

Start with $y(0) = 3$. Substituting this into the general solution gives:
$y(0) = 3 = Ae^0 + Be^0 = A + B$, or simply $3 = A + B$.

Currently you have **one** equation with **two** unknowns so more information is needed before you can determine *A* or *B.*

Next look at $y'(0) = 15$. First you need to differentiate the general solution: $y' = 3Ae^{3x} + 6Be^{6x}$. Substituting the initial condition into this equation gives: $y'(0) = 15 = 3Ae^{0} + 6Be^{0} = 3A + 6B$, or simply $15 = 3A + 6B$.

You now have **two** equations with **two** unknowns so you can use the method of solving **simultaneous equations** (see study guide: *Simultaneous Equations*) to solve these equations to get *A* and *B*. The equations:

$3 = A + B$ and $15 = 3A + 6B$ imply that $A = 1$ and $B = 2$.

When you have found any constants, substitute them into the general solution of your DE. You should now have a solution which doesn't contain any unknown constants. A solution like this one, where you have no unknowns, is called a **particular solution**.

In the above example you have:

$$y = e^{3x} + 2e^{6x}$$

is the **particular solution** of $y'' = 9y' - 18y$ when $y(0) = 3$ and $y'(0) = 15$.

Want to know more?

If you have any further questions about this topic you can make an appointment to see a **Learning Enhancement Tutor** in the **Student Support Service**, as well as speaking to your lecturer or adviser.

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