Learning Enhancement Team

Model Answers: Homogeneous Second Order Differential Equations

These are the model answers for the worksheet that has questions on homogeneous second order differential equations.

Homogeneous Second Order Differential Equations worksheet	Homogeneous Second Order Differential Equations study guide

1.

a. $$\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0$$

This second order ordinary differential equation **is** homogeneous because it contains only terms proportional to y and its derivatives. There are no independent constants or independent functions of x.

b. $$\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} = x$$

This second order ordinary differential equation is **not** homogeneous because it contains an independent function of x (the x on the right-hand side).

c. $$y'' + 2y = 0$$

This second order ordinary differential equation **is** homogeneous as it contains only terms proportional to y and its derivatives. There are no independent constants or independent functions of x.

d. $$\frac{d^2y}{dx^2} = \cos(x)y$$

This second order ordinary differential equation **is** homogeneous because it contains only terms proportional to *y* and its derivatives. There are no independent constants or independent functions of *x.*

e. $$y'' + y' = 4x$$

The above second order ordinary differential equation is **not** homogeneous because it contains an independent function of *x* (the 4*x* term on the right-hand side).

f. $$\frac{d^2y}{dx^2} + y = 4$$

The above second order ordinary differential equation is **not** homogeneous because it contains an independent constant term (the constant 4 on the right-hand side).

2.

a. $$y'' + x^2y' + y = 0$$

This second order ordinary differential equation **is** homogeneous because it contains only terms proportional to *y* and its derivatives. There are no independent constants or independent functions of *x.*

b. $$x\frac{dy}{dx} = \frac{d^2y}{dx^2} + 1$$

This second order ordinary differential equation is **not** homogeneous because it contains an independent constant term (the constant 1 on the right-hand side).

c. $$3y'' + x = y' - 1$$

This second order ordinary differential equation is **not** homogeneous for two reasons. It contains an independent function of *x* (the *x* term on the left-hand side). Also, it contains an independent constant term (the constant -1 on the right-hand side).

3.

a. $y'' - 3y' + 2y = 0$

You should first check whether this second order ordinary differential equation (ODE) is homogeneous:

There are no independent constants or independent functions of x, it contains **only terms proportional to *y* and its derivatives**. So this second order ODE **is** homogeneous.

Next, write down $y = e^{kx}$ (k is an unknown constant that you will solve) and find the derivatives y' and y'':

$y' = ke^{kx}$ and $y'' = k^2 e^{kx}$

You can then substitute these into the ODE to get:

$$k^2 e^{kx} - 3ke^{kx} + 2e^{kx} = 0$$

and e^{kx} is common to every term so you can factorise the equation:

$$e^{kx}\left(k^2 - 3k + 2\right) = 0$$

Since e^{kx} can never be zero, you can safely divide each side of the equation by e^{kx} without risk of losing information about a solution to the ODE. So you have:

$$k^2 - 3k + 2 = 0$$

This equation is now a **quadratic equation** for k (see study guides: *Solving Quadratic Equations by Factorisation* and *Solving Quadratic Equations using the Quadratic Formula*). This quadratic can be factorised:

$k^2 - 3k + 2 = 0$ becomes $(k-1)(k-2) = 0$

and so either

$k - 1 = 0$ or $k - 2 = 0$

so $k = 1$ and $k = 2$ are both solutions to this quadratic equation. These are **distinct real roots** so you should then substitute these values for k into $y = e^{kx}$ to get two solutions to the ODE:

$y_1 = e^{x}$ and $y_2 = e^{2x}$

To get the **general solution** you can multiply each of these by a constant and then add them together (this is possible because the differential equation is equal to zero when you substitute either solution into it, and zero plus zero is *still* zero.) So:

$y = Ae^{x} + Be^{2x}$ is the **general solution** to $y'' - 3y' + 2y = 0$.

(where *A* and *B* are constants).

b. $$y'' + 4y' + 4y = 0$$

First you should check whether this second order ordinary differential equation (ODE) is homogeneous:

There are no independent constants or independent functions of *x*, it contains **only terms proportional to *y* and its derivatives**. So this second order ODE **is** homogeneous.

Next, write down $y = e^{kx}$ (*k* is an unknown constant that you will solve) and find the derivatives y' and y'':

$$y' = ke^{kx} \qquad \text{and} \qquad y'' = k^2e^{kx}$$

You can then substitute these into the ODE to get:

$$k^2e^{kx} + 4ke^{kx} + 4e^{kx} = 0$$

and e^{kx} is common to every term so you can factorise the equation:

$$e^{kx}\left(k^2 + 4k + 4\right) = 0$$

Since e^{kx} can never be zero, you can divide each side of the equation by e^{kx} without risk of losing information about a solution to the ODE. So you have:

$$k^2 + 4k + 4 = 0$$

This equation is now a **quadratic equation** for *k* (see study guides: *Solving Quadratic Equations by Factorisation* and *Solving Quadratic Equations using the Quadratic Formula*). You can factorise the quadratic equation:

$$k^2 + 4k + 4 = 0 \qquad \text{becomes} \qquad (k+2)(k+2) = 0$$

and so $k + 2 = 0$, which means that $k = -2$. You have **one repeated root** for *k* so you can then substitute this value for *k* into $y = e^{kx}$ to get

$$y_1 = e^{-2x}$$

which is your first solution to the second order ODE. Since this is a second order differential equation, it will always have **two** solutions. When you have a **repeated real root** the second solution to the second order ordinary differential equation is found by multiplying the first solution by x (see study guide: *Homogeneous Second Order Differential Equations*).

So:

$y_2 = xy_1$ which means that $y_2 = xe^{-2x}$

To get the **general solution** you can multiply each of these by a constant and then add them together (this is possible because the differential equation is equal to zero when you substitute either solution into it, and zero plus zero is *still* zero.) So you get:

$$y = Ae^{-2x} + Bxe^{-2x} = e^{-2x}(A + Bx)$$

is the **general solution** to $y'' + 4y' + 4y = 0$ (where A and B are constants).

c. $$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 20y = 0$$

First you should check whether this second order ordinary differential equation (ODE) is homogeneous:

There are no independent constants or independent functions of x, it contains **only terms proportional to *y* and its derivatives**. So this second order ODE **is** homogeneous.

Next, write down $y = e^{kx}$ (k is an unknown constant that you will solve) and find the derivatives y' and y'':

$y' = ke^{kx}$ and $y'' = k^2e^{kx}$

You can then substitute these into the ODE to get:

$$k^2e^{kx} - 8ke^{kx} + 20e^{kx} = 0$$

and e^{kx} is common to every term so you can factorise the equation:

$$e^{kx}\left(k^2 - 8k + 20\right) = 0$$

Since e^{kx} can never be zero, you can divide each side of the equation by e^{kx} without risk of losing information about a solution to the ODE. So you have:

$$k^2 - 8k + 20 = 0$$

This equation is now a **quadratic equation** for *k* (see study guides: *Solving Quadratic Equations by Factorisation* and *Solving Quadratic Equations using the Quadratic Formula*).

You cannot easily factorise this equation so you can use the **Quadratic Formula** to solve for *k*. For a quadratic equation of the form:

$$ak^2 + bk + c = 0$$ the **Quadratic Formula** says: $$k = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In the equation $k^2 - 8k + 20 = 0$, $a = 1$, $b = -6$ and $c = 20$. So, substituting into the **Quadratic Formula** gives:

$$k = \frac{8 \pm \sqrt{64 - 80}}{2}$$ which can be simplified to: $$k = 4 \pm \frac{\sqrt{-16}}{2}$$

You have a square root of a negative number which means you can write this as a **complex number** (see study guide: *Basics of Complex Numbers*). First you can simplify a little more:

$$k = 4 \pm \frac{\sqrt{-16}}{2}$$ can be simplified to $$k = 4 \pm \frac{\sqrt{16}\sqrt{-1}}{2}$$

and so:

$$k = 4 \pm 2\sqrt{-1}$$

Using the laws of complex numbers, $\sqrt{-1} = \mathrm{i}$ (i is the **imaginary unit**). So your quadratic equation has two **complex solutions** for *k*:

$$k = 4 + 2\mathrm{i} \quad \text{and} \quad k = 4 - 2\mathrm{i}$$

When *k* has **complex roots** of the form $\alpha \pm \beta \mathrm{i}$, the **general solution** to your second order homogeneous differential equation has the form:

$$y = Ae^{\alpha x}\cos(\beta x) + Be^{\alpha x}\sin(\beta x)$$

where *A* and *B* are constants (see study guide: *Homogeneous Second Order Differential Equations*).

In this question you have:

$$k = 4 \pm 2\text{i}, \qquad \text{so } \alpha = 4 \text{ and } \beta = 2 \ .$$

You can substitute these into $y = Ae^{\alpha x}\cos(\beta x) + Be^{\alpha x}\sin(\beta x)$ to get:

$$y = Ae^{4x}\cos(2x) + Be^{4x}\sin(2x)$$

which is the **general solution** to $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 20y = 0$.

(You could also use the method from question 3 part (a), and substitute your values for k into $y = e^{kx}$ to get the two solutions to the second order ODE. You can then use Euler's Identity, see study guide: *Euler's Formula and De'Moivre's Theorem*, to reduce your solutions to the form of the general solution that was solved above.)

4.

a. $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$ with the initial conditions $y(0) = 0$ and $y'(0) = 1$

You should first check whether this second order ordinary differential equation (ODE) is homogeneous:

There are no independent constants or independent functions of x, it contains **only terms proportional to *y* and its derivatives**. So this second order ODE **is** homogeneous.

Next, write down $y = e^{kx}$ (k is an unknown constant that you will solve) and find the derivatives y' and y'':

$$y' = ke^{kx} \qquad \text{and} \qquad y'' = k^2e^{kx}$$

You can then substitute these into the ODE to get:

$$k^2e^{kx} + 3ke^{kx} + 2e^{kx} = 0$$

and e^{kx} is common to every term so you can factorise the equation:

$$e^{kx}\left(k^2 + 3k + 2\right) = 0$$

Since e^{kx} can never be zero, you can safely divide each side of the equation by e^{kx} without risk of losing information about a solution to the ODE. So you have:

$$k^2 + 3k + 2 = 0$$

This equation is now a **quadratic equation** for *k* (see study guides: *Solving Quadratic Equations by Factorisation* and *Solving Quadratic Equations using the Quadratic Formula*). This quadratic can be factorised:

$$k^2 + 3k + 2 = 0 \qquad \text{becomes} \qquad (k+1)(k+2) = 0$$

and so either

$$k + 1 = 0 \qquad \text{or} \qquad k + 2 = 0$$

so $k = -1$ and $k = -2$ are both solutions to this quadratic equation. These are **distinct real roots** so you should then substitute these values for *k* into $y = e^{kx}$ to get two solutions to the ODE:

$$y_1 = e^{-x} \qquad \text{and} \qquad y_2 = e^{-2x}$$

To get the **general solution** you can multiply each of these by a constant and then add them together (this is possible because the differential equation is equal to zero when you substitute either solution into it, and zero plus zero is *still* zero.) So:

$y = Ae^{-x} + Be^{-2x}$ is the **general solution** to $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$.

(where *A* and *B* are constants).

In order to find the **particular solution** you need to use the given **initial conditions**:

$y(0) = 0$ which tells you when $x = 0$ then $y = 0$

$y'(0) = 1$ which tells you when $x = 0$ then $y' = 1$.

You can substitute $y(0) = 0$ into the **general solution**:

$y = Ae^{-x} + Be^{-2x}$ which becomes $0 = A + B$ **(1)**

You now have **one** equation with **two** unknowns, so you need more information in order to find *A* and *B*. You can use the second initial condition $y'(0) = 1$ to get another equation in terms of *A* and *B*. In order to use this initial condition, you first need to differentiate the **general solution** to work out y'. So:

$$y' = -Ae^{-x} - 2Be^{-2x}$$

Now you can use the initial condition $y'(0) = 1$:

$y' = -Ae^{-x} - 2Be^{-2x}$ becomes $1 = -A - 2B$ **(2)**

So you now have **two** equations with **two** unknowns (the equations labelled as **(1)** and **(2)**. You can use these equations for A and B to find values for A and B. You can use the method for solving **simultaneous equations** (see study guide: *Simultaneous Equations*) to solve the equations:

$0 = A + B$ **(1)**

$1 = -A - 2B$ **(2)**.

If you subtract B from equation **(1)** you get:

$-B = A$

You can then substitute this into equation **(2)**:

$1 = -A - 2B$ becomes $1 = -(-B) - 2B$

which simplifies to:

$1 = -B$ so $B = -1$

Then substituting this value for B into the first equation gives:

$-(-1) = A$ so $A = 1$

You can then substitute these into the **general solution** to get:

$y = e^{-x} - e^{-2x}$ which is the **particular solution** to $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$

with initial conditions $y(0) = 0$ and $y'(0) = 1$.

b. $y'' + 6y' + 9y = 0$ with the boundary conditions $y(0) = 0$ and $y(1) = 2$

You should first check whether this second order ordinary differential equation (ODE) is homogeneous:

There are no independent constants or independent functions of x, it contains **only terms proportional to *y* and its derivatives**. So this second order ODE **is** homogeneous.

Next, write down $y = e^{kx}$ (k is an unknown constant that you will solve) and find the derivatives y' and y'':

$$y' = ke^{kx} \qquad \text{and} \qquad y'' = k^2e^{kx}$$

You can then substitute these into the ODE to get:

$$k^2e^{kx} + 6ke^{kx} + 9e^{kx} = 0$$

and e^{kx} is common to every term so you can factorise the equation:

$$e^{kx}\left(k^2 + 6k + 9\right) = 0$$

Since e^{kx} can never be zero, you can safely divide each side of the equation by e^{kx} without risk of losing information about a solution to the ODE. So you have:

$$k^2 + 6k + 9 = 0$$

This equation is now a **quadratic equation** for *k* (see study guides: *Solving Quadratic Equations by Factorisation* and *Solving Quadratic Equations using the Quadratic Formula*). This quadratic can be factorised:

$$k^2 + 6k + 9 = 0 \qquad \text{becomes} \qquad (k+3)(k+3) = 0$$

and so:

$$k + 3 = 0$$

so $k = -3$ is a repeated solution to this quadratic equation. This is a **repeated real root** so you should substitute this value for *k* into $y = e^{kx}$ to get your first solution to the ODE:

$$y_1 = xe^{-3x}$$

When you have a **repeated real root** the second solution to the second order ordinary differential equation is found by multiplying the first solution by *x* (see study guide: *Homogeneous Second Order Differential Equations*). So:

$$y_2 = xy_1 \qquad \text{which means that} \qquad y_2 = xe^{-3x}$$

To find the **general solution** you can multiply each of these by a constant and then add them together (this is possible because the differential equation is equal to zero when you substitute either solution into it, and zero plus zero is *still* zero.) So you get:

$$y = Ae^{-3x} + Bxe^{-3x} = e^{-3x}(A + Bx)$$

is the **general solution** to $y'' + 6y' + 9y = 0$ (where *A* and *B* are constants).

Now that you know the **general solution**, you can use the boundary conditions to find the **particular solution**. You have:

$y(0)=0$ which tells you when $x=0$ then $y=0$

$y(1)=2$ which tells us when $x=1$ then $y=2$.

You can substitute $y(0)=0$ into the **general solution**:

$$y=e^{-3x}(A+Bx) \quad \text{which becomes} \quad 0=A$$

You now know that $A=0$, but you need more information in order to find *B*. You can use the second boundary condition $y(1)=2$ to get another equation in terms of *B*. Your solution now looks like:

$$y=Bxe^{-3x}$$

since $A=0$. Then you can substitute $y(1)=2$ into this to get:

$$2=Be^{-3}$$

and you can divide both sides by e^{-3} to get:

$$2e^{3}=B$$

Substituting this into your solution gives:

$$y=2e^{3}xe^{-3x} \quad \text{and this simplifies to} \quad y=2xe^{3-3x}$$

which is the **particular solution** to $y''+6y'+9y=0$ with the boundary conditions $y(0)=0$ and $y(1)=2$.