Learning Enhancement Team

University of East Anglia
STUDENT SUPPORT SERVICE

Steps into Differential Equations

Homogeneous First Order Differential Equations

This guide helps you to identify and solve homogeneous first order ordinary differential equations.

Introduction

A **differential equation** (or **DE**) is any equation which contains derivatives, see study guide: *Basics of Differential Equations*. To make the best use of this guide you will need to be familiar with some of the terms used to categorise differential equations.

Linear DE: The function *y* and any of its derivatives can **only** be multiplied by a constant or a function of *x*.

Nonlinear DE: More complicated functions of *y* and its derivatives appear **as well as** multiplication by a constant or a function of *x*.

Ordinary differential equation (ODE): Contains only ordinary derivatives.

First order DE: Highest order derivative is first order: $\frac{dy}{dx}$ or y' or $\dot{x}$.

Top tip: You may see the term **homogeneous** used to describe differential equations of higher order, especially when you are identifying and solving **second order linear differential equations**. For more information see study guide: *Homogeneous Second Order Differential Equations*.

However, there is an entirely different meaning for a **homogeneous** first order ordinary differential equation. This guide is only concerned with, and **the following method is only applicable to**, first order ODEs. The examples that follow will concern a variable *y* which is itself a function of a variable *x*.

A first order ODE is called **homogeneous** if the DE remains unchanged when you replace *y* with qy and *x* with qx, where q is a constant. In other words you can make these substitutions and all the *q*'s cancel.

To identify a homogeneous first order ODE:

1. Replace y with qy and x with qx in the ODE.
2. Use algebra to simplify the new ODE
3. You have a homogeneous first order ODE only if all the q's cancel.

As you shall see, integration is the most powerful tool at your disposal for solving homogeneous first order ODEs. Therefore you should be comfortable with the basics of integration if you want to be able to solve them. You can use the resources: *Steps into Calculus* to help you with your integration and differentiation skills.

Example: Which of these first order ordinary differential equations are homogeneous?

(a) $\frac{dy}{dx} = 2xy$ (b) $y\frac{dy}{dx} = 2x$ (c) $\frac{dy}{dx} = \frac{3x^2 + y^2}{xy}$

Solutions:

(a) Replace y with qy and x with qx in the ODE to get:

$$\frac{dy}{dx} = 2q^2xy$$

As you still have q in the ODE this is not a homogeneous ODE. In fact it is a first order separable ODE and you can use the **separation of variables** method to solve it, see study guide: *Separable Differential Equations*.

(b) Replace y with qy and x with qx in the ODE to get:

$$qy\frac{dy}{dx} = 2qx$$

You can divide both sides by q to recover the original ODE and therefore this is a homogeneous ODE. It is also a first order separable ODE.

You often get DEs that can be categorised as more than one type. Therefore you have to decide which method to use to solve the DE. The answer is easy: you should use the simplest method. Separation of variables is more straightforward than the method for solving homogeneous DEs described below and so:

If your DE is **both separable and homogeneous** then use **separation of variables** to solve it

(c) Replace y with qy and x with qx in the ODE to get:

$$\frac{dy}{dx} = \frac{3q^2x^2 + q^2y^2}{q^2xy}$$

You can factorise the numerator to see that q^2 cancels down, leaving the original ODE. This time you cannot separate the variables so a new method is required.

Solving homogeneous first order ODEs

There is a method to solve homogeneous first order ODEs (which are not separable).

Step 1: Check your ODE is homogeneous and **not** separable.

Step 2: Make the substitutions:

$$\boxed{y = ux} \qquad \text{and} \qquad \boxed{\frac{dy}{dx} = u + x\frac{du}{dx}}$$

The second of these results is found by differentiating the first result using the product rule (see study guide: *The Product Rule*).

This eliminates y from your ODE. Making this substitution in a homogeneous ODE always makes a new ODE with variables u and x which is **separable**.

Step 3: Solve your new ODE using separation of variables.

Step 4: Replace u using $u = \frac{y}{x}$.

Step 5: Check your solution by differentiation.

Example: Solve $\frac{dy}{dx} - \frac{y}{x} = e^{\frac{y}{x}}$.

Step 1: Replacing y with qy and x with qx in the ODE you get:

$$\frac{dy}{dx} - \frac{qy}{qx} = e^{\frac{qy}{qx}}$$

You can cancel all the q's so the ODE is homogeneous. Furthermore you cannot separate the variables here.

Step 2: Next make the substitutions $y = ux$ and $\frac{dy}{dx} = u + x\frac{du}{dx}$.

$$\frac{dy}{dx} - \frac{y}{x} = e^{\frac{y}{x}} \qquad \text{becomes} \qquad u + x\frac{du}{dx} - u = e^{u}$$

Here u cancels on the left-hand side to give: $x\frac{du}{dx} = e^{u}$

Step 3: You can now separate the variables to give:

$$e^{-u}du = \frac{1}{x}dx$$

and integrating both sides gives: $-e^{-u} = \ln x + c$

where the constant c results from collecting together the two integration constants from the indefinite integration of both sides of the equation.

Step 4: Replacing u using $u = \frac{y}{x}$ gives:

$$-e^{-\frac{y}{x}} = \ln(x) + c$$ which is the **general solution** of $\frac{dy}{dx} - \frac{y}{x} = e^{\frac{y}{x}}$

Step 5: You can check your general solution by using differentiation. Here your general solution is best differentiated by using **implicit differentiation** (see study guides: *The Differential Operator* and *Implicit Differentiation*) which gives:

$$-e^{\frac{y}{x}}\left(\frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2}\right) = \frac{1}{x}$$

This can be rearranged to give the original ODE $\frac{dy}{dx} - \frac{y}{x} = e^{\frac{y}{x}}$

Example: Solve $\frac{dy}{dx} = \frac{3x^2 + y^2}{xy}$ with the boundary condition $y(1) = 1$.

Step 1: Example (c) on page 2 of this guide shows you that this is a homogeneous differential equation. It is worth noticing that the equation can be rewritten as:

$$\frac{dy}{dx} = \frac{3x}{y} + \frac{y}{x}$$

Identifying either x/y or y/x in a homogenous ODE can help you with the next step which involves substitution.

Step 2: Next make the substitutions $y = ux$ and $\frac{dy}{dx} = u + x\frac{du}{dx}$.

$$\frac{dy}{dx} = \frac{3x}{y} + \frac{y}{x} \qquad \text{becomes} \qquad u + x\frac{du}{dx} = \frac{3}{u} + u$$

You can subtract *u* from each side to give: $x\frac{du}{dx} = \frac{3}{u}$

Step 3: You can now separate the variables to give:

$$udu = \frac{3}{x}dx$$

and integrating both sides gives: $\frac{u^2}{2} = 3\ln x + c$

where the constant *c* results from collecting together the two integration constants from the indefinite integration of both sides of the equation.

Step 4: Replacing *u* using $u = \frac{y}{x}$ gives:

$$\frac{y^2}{2x^2} = 3\ln x + c$$ which is the **general solution** of $\frac{dy}{dx} = \frac{3x^2 + y^2}{xy}$

Step 5: The general solution of this equation is best differentiated using implicit differentiation, which gives:

$$\frac{y}{x^2}\frac{dy}{dx} - \frac{y^2}{x^3} = \frac{3}{x}$$ which can be rearranged to give the original ODE.

Remember that *y* is a function of *x* so you could write $y(x)$. Then $y(1) = 1$ tells you that $y = 0$ when $x = 2$. This is the **boundary condition**. Using this in the general solution tells you that:

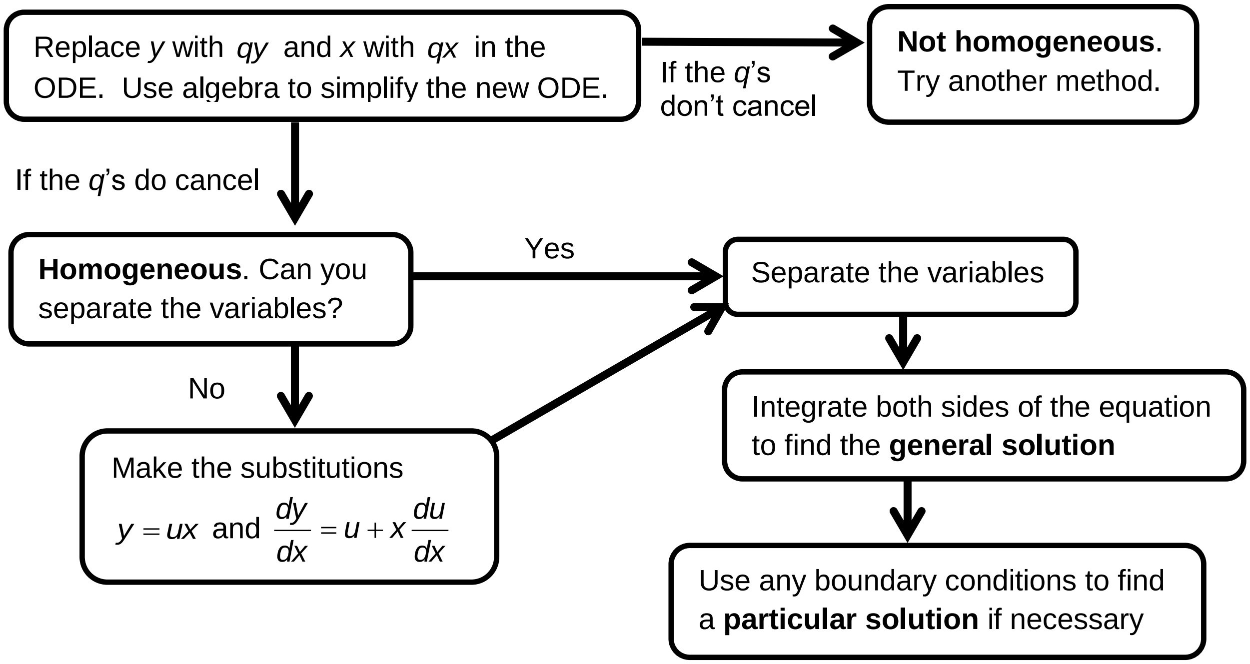
$$\frac{y^2}{2x^2} = 3\ln(x) + c \qquad \text{becomes} \qquad \frac{1}{2} = 0 + c$$

Or in other words, $c = 1/2$. You can now substitute this value back into your general solution to give a solution particular to the given boundary conditions:

$$\frac{y^2}{2x^2} = 3\ln(x) + \frac{1}{2}$$ **particular solution** of $\frac{dy}{dx} = \frac{3x^2 + y^2}{xy}$ **when** $y(1) = 1$

A solution like this one, where you have no unknowns, is called a **particular solution**.

Flow chart for solving homogeneous first order ODEs



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