Learning Enhancement Team

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Model Answers: Homogeneous First Order Differential Equations

These are the model answers for the worksheet that has questions on homogeneous first order differential equations.

Homogeneous First Order Differential Equations worksheet	Homogeneous First Order Differential Equations study guide

1.

a. $\frac{dy}{dx} = xy$ is **not** homogeneous.

You can replace *x* with qx and *y* with qy in the ordinary differential equation (ODE) to get:

$$\frac{dy}{dx} = q^2 xy$$

As you cannot cancel *q* from this ordinary differential equation, this is **not** a homogeneous first order ordinary differential equation.

b. $y\frac{dy}{dx} = 4x$ **is** homogeneous.

You can replace *x* with qx and *y* with qy in the first order ordinary differential equation to give:

$$qy\frac{dy}{dx} = 4qx$$

Then you can divide each side of the equation by *q* to recover the original ordinary differential equation (ODE), so this **is** a homogeneous first order ODE.

c. $y' + \frac{2y^2}{x} = 0$ is **not** homogeneous.

If you replace x with qx and y with qy, the ordinary differential equation becomes:

$y' + \frac{2q^2y^2}{qx} = 0$ which simplifies to $y' + \frac{2qy^2}{x} = 0$

You cannot cancel q from this equation, so this is **not** a homogeneous first order ordinary differential equation.

d. $\frac{dy}{dx} = \frac{x^2 + y^2}{4xy}$ **is** homogeneous.

You can replace x with qx and y with qy in the ordinary differential equation (ODE) to get:

$$\frac{dy}{dx} = \frac{q^2x^2 + q^2y^2}{4q^2xy}$$

Then you can factorise the numerator to get:

$\frac{dy}{dx} = \frac{q^2(x^2 + y^2)}{4q^2xy}$ which simplifies to $\frac{dy}{dx} = \frac{(x^2 + y^2)}{4xy}$

so the q^2 term has cancelled, leaving the original ODE. So this **is** a homogeneous first order ordinary differential equation.

e. $x^2 + yy' = 4x$ is **not** homogeneous.

If you replace x with qx and y with qy in the ordinary differential equation (ODE), you get:

$q^2x^2 + qyy' = 4qx$ which can be simplified to $qx^2 + yy' = 4x$

As you still have q in the ODE, this is **not** a homogeneous first order ordinary differential equation.

f. $x^2y' + 2xy = 0$ **is** homogeneous.

You can replace x with qx and y with qy in the differential equation to give:

$$q^2x^2y' + 2q^2xy = 0$$

Then you can divide each side of the equation by q^2 to get:

$$x^2y' + 2xy = 0$$

All terms involving q have been removed and you are now back to the original equation. So this **is** a homogeneous first order ordinary differential equation.

2.

a. $$x^2\frac{dy}{dx} = 2xy$$

To show that this ordinary differential equation (ODE) is homogeneous you can use the same method used in the solutions to question 1:

If you replace x with qx and y with qy in the ODE you get:

$$q^2x^2\frac{dy}{dx} = 2q^2xy$$

You can then divide both sides of the equation by q^2 to get:

$$x^2\frac{dy}{dx} = 2xy$$

All the q's have cancelled and this is the original ODE, so this first order ordinary differential equation **is** homogeneous.

b. $$\frac{dy}{dx} = \frac{y^2 + 4xy}{x^2}$$

To show that this ordinary differential equation (ODE) is homogeneous you can replace x with qx and y with qy in the ODE to get:

$$\frac{dy}{dx} = \frac{q^2y^2 + 4q^2xy}{q^2x^2}$$

You can then factorise the numerator:

$$\frac{dy}{dx} = \frac{q^2(y^2 + 4xy)}{q^2x^2}$$ and the q^2 terms cancel so this simplifies to:

$$\frac{dy}{dx} = \frac{y^2 + 4xy}{x^2}$$ which is the original ODE.

Since all the terms involving q have been cancelled, this first order ordinary differential equation is homogeneous.

c. $x^2yy' = x^3 + y^3$

To show that this ordinary differential equation (ODE) is homogeneous you can replace x with qx and y with qy in the ODE to get:

$$q^3x^2yy' = q^3(x^3 + y^3)$$

You can then divide each side of the equation by q^3 to recover the original ODE:

$$x^2yy' = x^3 + y^3$$

Since all terms involving q have been cancelled, this first order ordinary differential equation **is** homogeneous.

3.

a. $x\frac{dy}{dx} = 2y$

First you should check whether this ordinary differential equation (ODE) is homogeneous. You can replace x with qx and y with qy in the ODE to give:

$$qx\frac{dy}{dx} = 2qy$$

Then you can divide both sides by q to recover the original ODE:

$x\frac{dy}{dx} = 2y$ so this first order ODE **is** homogeneous.

In fact, this is a **separable first order differential equation**. If your differential equation is both **separable** and **homogeneous** then use **separation of variables** to solve it as this is generally easier (see study guide: *Separable Differential Equations*).

To solve using the method of **separation of variables**, first you need to separate the variables in the ODE. You can divide each side by x and then divide each side by y to get:

$$\frac{1}{y}\frac{dy}{dx} = \frac{2}{x}$$

You can then multiply both sides by dx to get:

$$\frac{dy}{y} = \frac{2dx}{x}$$

The variables have now been separated, with all functions concerning *y* on the left-hand side and all functions of *x,* including any constants, on the right-hand side.

The next step is to integrate both sides:

$$\int \frac{1}{y} dy = \int \frac{2}{x} dx$$

which after doing the integration becomes:

$$\ln(y) = 2\ln(x) + c$$

where *c* is the constant of integration.

Then using the **logarithmic transformation** (see study guide: *Basics of Logarithms*) you get:

$$e^{\ln(y)} = e^{2\ln(x)+c} \qquad \text{which becomes} \qquad e^{\ln(y)} = e^{\ln(x^2)}e^c$$

since $2\ln(x) = \ln(x^2)$, and $e^{a+b} = e^a e^b$ (for some arbitrary *a* and *b*). Then since Euler's function e^x is the **inverse** of the natural logarithm $\ln(x)$,

$$e^{\ln(y)} = e^{\ln(x^2)}e^c \qquad \text{can be written as:} \qquad y = Ax^2 \qquad (\text{where } A = e^c).$$

So $y = Ax^2$ is the **general solution** to $x\frac{dy}{dx} = 2y$.

You should check this by differentiating the solution and substituting it back into the original differential equation.

b. $$\frac{dy}{dx} = \frac{2x^2 + y^2}{xy}$$

First you should check whether this ordinary differential equation (ODE) is homogeneous. You can replace *x* with qx and y with qy in the ODE to give:

$$\frac{dy}{dx} = \frac{2q^2x^2 + q^2y^2}{q^2xy}$$

You can then factorise the numerator to get:

$$\frac{dy}{dx} = \frac{q^2(2x^2 + y^2)}{q^2xy} \qquad \text{which simplifies to} \qquad \frac{dy}{dx} = \frac{2x^2 + y^2}{xy}$$

so the ODE is homogeneous. This time the ODE is **not** separable so you need to use another method.

The first order ODE can be rewritten as:

$$\frac{dy}{dx} = \frac{2x^2}{xy} + \frac{y^2}{xy} \qquad \text{which simplifies to} \qquad \frac{dy}{dx} = \frac{2x}{y} + \frac{y}{x}$$

Then you can make the substitutions:

$$y = ux \qquad \text{and} \qquad \frac{dy}{dx} = u + x\frac{du}{dx} \qquad \text{(where } u \text{ is a function of } x\text{)}$$

(the second of these is found by differentiating $y = ux$ using the **product rule**, see study guide: *The Product Rule*).

Substituting these into $\frac{dy}{dx} = \frac{2x}{y} + \frac{y}{x}$ gives:

$$u + x\frac{du}{dx} = \frac{2x}{ux} + \frac{ux}{x} \qquad \text{which simplifies to} \qquad u + x\frac{du}{dx} = \frac{2}{u} + u$$

This new ODE in terms of *u* and *x* should be **separable** (see study guides: *Homogeneous First Order Differential Equations* and *Separable Differential Equations*). You can subtract *u* from each side to get:

$$x\frac{du}{dx} = \frac{2}{u}$$

Then you can divide each side by x and multiply each side by *u* to get:

$$u\frac{du}{dx} = \frac{2}{x}$$

Then multiply both sides by *dx* to get:

$$udu = 2\frac{dx}{x}$$

The variables have now been separated, with all functions concerning *y* on the left-hand side and all functions of *x,* including any constants, on the right-hand side. The next step is to integrate both sides:

$$\int udu = 2\int \frac{1}{x}dx$$

which after integrating becomes:

$$\frac{u^2}{2} = 2\ln(x) + c$$

where *c* is the constant resulting from indefinite integration. You can then multiply each side of the equation by 2 and take the square root of both sides to get:

$$u = \sqrt{4\ln(x) + 2c}$$

You then need to replace *u,* using the substitution $u = \frac{y}{x}$ (as $y = ux$ from earlier):

$$\frac{y}{x} = \sqrt{4\ln(x) + 2c}$$

and then you can multiply each side by *x* to get:

$$y = x\sqrt{4\ln(x) + 2c}$$

Finally, if you write $c = 2k$ where *k* is a constant (so $2c = 4k$), then you can factorise the terms inside the square root to get:

$$y = x\sqrt{4(\ln(x) + k)}$$

which can be simplified to:

$$y = x\sqrt{4}\sqrt{\ln(x) + k}$$

and since $\sqrt{4} = 2$ you have:

$$y = 2x\sqrt{\ln(x) + k}$$

which is the **general solution** to $\frac{dy}{dx} = \frac{2x^2 + y^2}{xy}$.

You should check this by differentiating the solution and substituting it back into the original differential equation.

c. $$y' - \frac{y}{x} = 1$$

First you should check whether this ordinary differential equation (ODE) is homogeneous. You can replace *x* with qx and y with qy in the ODE to give:

$y' - \frac{qy}{qx} = 1$ which simplifies to $y' - \frac{y}{x} = 1$

which is the original ODE, so this first order ordinary differential equation **is** homogeneous.

The ODE is **not** separable, so you need the method used in the solution to question 3 part (b).

You can make the substitutions:

$$y = ux \qquad \text{and} \qquad \frac{dy}{dx} = u + x\frac{du}{dx} \qquad \text{(where } u \text{ is a function of } x\text{)}$$

(the second of these is found by differentiating $y = ux$ using the **product rule**, see study guide: *The Product Rule*).

Substituting these into $y' - \frac{y}{x} = 1$ gives:

$$u + x\frac{du}{dx} - \frac{ux}{x} = 1$$

which simplifies to:

$$u + x\frac{du}{dx} - u = 1$$

and the *u* terms cancel out, so:

$$x\frac{du}{dx} = 1$$

This is a **separable** ODE in terms of *u* and *x* (see study guides: *Homogeneous First Order Differential Equations* and *Separable Differential Equations*). You can divide both sides by *x* and then multiply each side by *dx* to get:

$$du = \frac{dx}{x}$$

The next step is to integrate both sides:

$$\int du = \int \frac{1}{x} dx$$

which after integrating becomes:

$$u = \ln(x) + c$$

where c is the constant resulting from indefinite integration. You can then replace u, using the substitution $u = \frac{y}{x}$ (since $y = ux$ from earlier), to get:

$$\frac{y}{x} = \ln(x) + c$$

and then you can multiply each side by x to get:

$$y = x(\ln(x) + c)$$

which is the **general solution** to $y' - \frac{y}{x} = 1$.

You should check this by differentiating the solution and substituting it back into the original differential equation.

4.

a. $$xy\frac{dy}{dx} - 4x^2 - y^2 = 0 \qquad \text{where } y(1) = 7$$

You are given that this first order ordinary differential equation (ODE) **is** homogeneous, so there is no need to check. You cannot separate the variables so the ODE is **not** separable.

You can make the substitutions:

$$y = ux \qquad \text{and} \qquad \frac{dy}{dx} = u + x\frac{du}{dx} \qquad \text{(where } u \text{ is a function of } x\text{)}$$

(the second of these is found by differentiating $y = ux$ using the **product rule**, see study guide: *The Product Rule*).

Substituting these into $xy\frac{dy}{dx} - 4x^2 - y^2 = 0$ gives:

$$ux^2\left(x\frac{du}{dx} + u\right) - 4x^2 - u^2x^2 = 0$$

You can then divide through by x^2 since this is common to all terms in the equation:

$$u\left(x\frac{du}{dx} + u\right) - 4 - u^2 = 0$$

Next you can multiply out the brackets on the left:

$$ux\frac{du}{dx} + u^2 - 4 - u^2 = 0$$

Here, the u^2 terms cancel out, so you can simplify the ODE further:

$$ux\frac{du}{dx} - 4 = 0$$

This first order ODE is now **separable**. You can add 4 to both sides and then divide both sides by *x* to get:

$$u\frac{du}{dx} = \frac{4}{x}$$

You can then multiply both sides by *dx* to get:

$$udu = \frac{4dx}{x}$$

The variables have now been separated, with all functions concerning *y* on the left-hand side and all functions of *x,* including any constants, on the right-hand side. The next step is to integrate both sides:

$$\int udu = \int \frac{4}{x}dx$$

which after integrating becomes

$$\frac{u^2}{2} = 4\ln(x) + c$$

where *c* is the constant resulting from indefinite integration. You can multiply both sides by 2 and then take the square root of each side to get:

$$u = \sqrt{8\ln(x) + 2c}$$

Then you can replace u using the substitution $u = \frac{y}{x}$ (since $y = ux$ from earlier):

$$\frac{y}{x} = \sqrt{8\ln(x) + 2c}$$

and multiply each side by *x* to get:

$$y = x\sqrt{8\ln(x) + 2c}$$

if you write $c = 2k$, where *k* is a constant (so $2c = 4k$), then you can factorise the terms inside the square root to get:

$$y = x\sqrt{4(2\ln(x) + k)}$$ which simplifies to $$y = 2x\sqrt{(2\ln(x) + k)}$$

and since $2\ln(x) = \ln(x^2)$, this simplifies to

$$y = 2x\sqrt{\ln(x^2) + k}$$

the **general solution** to $xy\frac{dy}{dx} - 4x^2 - y^2 = 0$.

In order to find the **particular solution** you need to use the boundary condition

$y(1) = 7$ which tells you that when $x = 1$ then $y = 7$.

You can substitute this into the **general solution**:

$y = 2x\sqrt{\ln(x^2) + k}$ becomes $7 = 2\sqrt{\ln(1) + k}$

and since $\ln(1) = 0$ you have $7 = 2\sqrt{k}$. Then you can square both sides to get: $k = 49/4 = 12.25$

So:

$$y = 2x\sqrt{\ln(x^2) + 12.25}$$

is the **particular solution** to $xy\frac{dy}{dx} - 4x^2 - y^2 = 0$ when $y(1) = 7$.

You should check this by differentiating the solution and substituting it back into the original differential equation.

b. $x\frac{dy}{dx} = y + x$ where $y(1) = 2$

You are given that this first order ordinary differential equation (ODE) **is** homogeneous, so there is no need to check. You cannot separate the variables so the ODE is **not** separable.

You can make the substitutions:

$y = ux$ and $\frac{dy}{dx} = u + x\frac{du}{dx}$ (where *u* is a function of *x*)

(the second of these is found by differentiating $y = ux$ using the **product rule**, see study guide: *The Product Rule*).

Substituting these into $x\frac{dy}{dx} = y + x$ gives:

$$x\left(u + x\frac{du}{dx}\right) = ux + x$$

You can then divide both sides by *x,* since *x* is common to each term, to get:

$$u + x\frac{du}{dx} = u + 1$$

Next you can subtract u from each side of the equation to get:

$$x\frac{du}{dx} = 1$$

This equation is now **separable**. You can divide both sides by *x* and then multiply each side by *dx* to get:

$$du = \frac{dx}{x}$$

The variables have now been separated, with all functions concerning *y* on the left-hand side and all functions of *x,* including any constants, on the right-hand side. The next step is to integrate both sides:

$$\int du = \int \frac{1}{x} dx$$

which after integrating becomes

$$u = \ln(x) + c$$

where *c* is the constant resulting from indefinite integration. Then you can replace *u* using the substitution $u = \frac{y}{x}$ (since $y = ux$ from earlier) to get:

$$\frac{y}{x} = \ln(x) + c$$

and then multiply through by *x* to give:

$$y = x(\ln(x) + c)$$

which is the **general solution** to $x\frac{dy}{dx} = y + x$.

In order to find the **particular solution** you need to use the boundary condition

$y(1) = 2$ which tells you that when $x = 1$ then $y = 2$.

You can then substitute this into the **general solution**:

$$y = x(\ln(x) + c) \qquad \text{becomes} \qquad 2 = \ln(1) + c$$

so $c = 2$, and therefore

$$y = x(\ln(x) + 2)$$

is the **particular solution** of $x\dfrac{dy}{dx} = y + x$ when $y(1) = 2$.

You should check this by differentiating the solution and substituting it back into the original differential equation.