Learning Enhancement Team

University of East Anglia
STUDENT SUPPORT SERVICE

Steps into Trigonometry

Further Trigonometry

This guide extends the basic ideas of trigonometry by introducing the reciprocal trigonometric functions (secant, cosecant and cotangent) and some fundamental trigonometric identities.

Introduction

The study guide: *Trigonometric Ratios: Sine, Cosine and Tangent* introduced the idea that you can use ratios of the sides of a right-angled triangle to find out properties of its angles. These fundamental functions are called **sine**, **cosine** and **tangent**. Specifically for a right-angled triangle:

$$\text{sine of an angle} = \frac{\text{length of side opposite to the angle}}{\text{length of the hypotenuse}}$$

$$\text{cosine of an angle} = \frac{\text{length of side adjacent to the angle}}{\text{length of the hypotenuse}}$$

$$\text{tangent of an angle} = \frac{\text{length of side opposite to the angle}}{\text{length of side adjacent to the angle}}$$

You write these as $\sin\theta$, $\cos\theta$ and $\tan\theta$ respectively for an angle θ (theta). These ideas were extended to all values of an angle in the study guide: *Trigonometric Functions*.

Trigonometric functions and the laws of indices

The basic trigonometric functions play a role in almost every part of mathematics. They follow interesting patterns and hold the key to unlocking many problems. In their manipulation it is often necessary to raise them to a positive power (square or cube for example) and it is useful to know how to write such an operation (if you need help with powers and indices read the study guide: *Laws of Indices* before continuing on). Let's consider the square of $\sin\theta$. In the same ways that the square of *x* means $x \cdot x$, the square of $\sin\theta$ means $\sin\theta \cdot \sin\theta$. However there is a difference in the way this is written as a power. In algebra $x \cdot x$ is written x^2 but $\sin\theta \cdot \sin\theta$ is written $\sin^2\theta$ **not** $\sin\theta^2$. You can continue in a similar manner and write any positive, whole number power of *x* for sine, cosine or tangent.

For any positive, whole number power *n*:

$(\sin\theta)^n$	is written	$\sin^n\theta$
$(\cos\theta)^n$	is written	$\cos^n\theta$
$(\tan\theta)^n$	is written	$\tan^n\theta$

The guide also introduced the inverses of the trigonometric ratios: $\sin^{-1}\theta$, $\cos^{-1}\theta$ and $\tan^{-1}\theta$. In the laws of indices negative powers imply **reciprocals**, where *x* is underneath a dividing line, however in function theory and superscript "-1" implies the **inverse** of a function (see study guide: *Inverse Functions and Graphs*). Understandably, this often leads to confusion. **In trigonometry the superscript** -1 **means the inverse**. In order to clarify further, the reciprocals of the trigonometric ratios are given special names in their own right:

$$\text{cosecant of an angle} = \frac{1}{\text{sine of an angle}} = \frac{\text{length of the hypotenuse}}{\text{length of side opposite to the angle}}$$

$$\text{secant of an angle} = \frac{1}{\text{cosine of an angle}} = \frac{\text{length of the hypotenuse}}{\text{length of side adjacent to the angle}}$$

$$\text{cotangent of an angle} = \frac{1}{\text{tangent of an angle}} = \frac{\text{length of side adjacent to the angle}}{\text{length of side opposite to the angle}}$$

The **cosecant** of an angle θ is written as either $\csc\theta$ or $\operatorname{cosec}\theta$: $\csc\theta = \dfrac{1}{\sin\theta}$

The **secant** of an angle θ is written as $\sec\theta$: $\sec\theta = \dfrac{1}{\cos\theta}$

The **cotangent** of an angle θ is written $\cot\theta$: $\cot\theta = \dfrac{1}{\tan\theta}$

If you have some mathematics which involves the reciprocal of a trigonometric ratio it is good practice to use cosecant, secant or cotangent instead. Similarly you can use the reciprocal forms of sine cosine and tangent to help you think about questions involving cosecant, secant and cotangent. Many question involve simplifying trigonometric formulas in some way and a good tactic is to change everything into sines and cosines to see what you are dealing with.

Example: Simplify $\cos\theta\sec\theta+\sin\theta\csc\theta$.

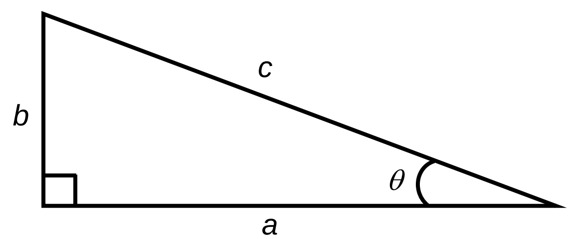
On first inspection this question looks difficult. However, the definitions of $\sec\theta$ and $\csc\theta$ on the previous page tell you that:

$$\cos\theta\frac{1}{\cos\theta}+\sin\theta\frac{1}{\sin\theta}=\frac{\cos\theta}{\cos\theta}+\frac{\sin\theta}{\sin\theta}=1+1=2$$

This idea will be extended in the next section.

Some fundamental trigonometric identities

The trigonometric functions combine in surprising ways to give extremely useful are far reaching results. The results that are derived in this section give some insight into how mathematicians think and how they create new and useful mathematics. You should follow the derivation of each result carefully. They will use the definitions introduced in this guide and the basic properties of the triangle:



1. Deriving the important relationship $\tan\theta=\dfrac{\sin\theta}{\cos\theta}$.

Firstly satisfy yourself that:

$$\sin\theta=\frac{b}{c} \quad \text{which rearranges to} \quad b=c\sin\theta$$

$$\cos\theta=\frac{a}{c} \quad \text{which rearranges to} \quad a=c\cos\theta$$

Now recognise that $\tan\theta=\dfrac{b}{a}$ and using the two results above:

$$\tan\theta=\frac{c\sin\theta}{c\cos\theta} \quad \text{cancelling down the } c\text{'s gives} \quad \boxed{\tan\theta=\frac{\sin\theta}{\cos\theta}}$$

Which is the required result. This tells you that **the tangent of any angle is the sine of that angle divided by its cosine**.

Example: Show that $1-\tan^2\theta = (\cos^2\theta - \sin^2\theta)\sec^2\theta$.

Many problems involving trigonometric identities begin by asking you to **show** a given piece of mathematics. This means you have to confirm that result is correct. To do this, pick either the left- or right-hand side of the equals sign and manipulate the mathematics to show it is equal to the other side. Often you should choose the left-hand side. In the example above first use the identity for $\tan\theta$:

$$1-\tan^2\theta = 1-\frac{\sin^2\theta}{\cos^2\theta}$$

Next, write this result as a single fraction:

$$1-\frac{\sin^2\theta}{\cos^2\theta} = \frac{\cos^2\theta}{\cos^2\theta} - \frac{\sin^2\theta}{\cos^2\theta} = \frac{\cos^2\theta - \sin^2\theta}{\cos^2\theta}$$

Finally use the definition of $\sec\theta$:

$$\frac{\cos^2\theta - \sin^2\theta}{\cos^2\theta} = (\cos^2\theta - \sin^2\theta)\frac{1}{\cos^2\theta} = (\cos^2\theta - \sin^2\theta)\sec^2\theta$$

as required. You have shown that $1-\tan^2\theta = (\cos^2\theta - \sin^2\theta)\sec^2\theta$ and so the question is answered.

2. Deriving the important relationship $\cos^2\theta + \sin^2\theta = 1$.

Pythagoras' theorem tells you that (for the triangle on the previous page):

$$a^2 + b^2 = c^2$$ (see study guide: *Pythagoras' Theorem*)

As you saw in the previous derivation, $a = c\cos\theta$ and $b = c\sin\theta$ so:

$$a^2 = (c\cos\theta)^2 = c^2\cos^2\theta$$
$$b^2 = (c\sin\theta)^2 = c^2\sin^2\theta$$

Substituting these into $a^2 + b^2 = c^2$ gives:

$c^2\cos^2\theta + c^2\sin^2\theta = c^2$ and dividing by c^2 shows $\boxed{\cos^2\theta + \sin^2\theta = 1}$

Example: Show that $(\cos\theta + \sin\theta)^2 = 1 + 2\cos\theta\sin\theta$.

Open the brackets on the left-hand side to find:

$$(\cos\theta + \sin\theta)^2 = \cos^2\theta + 2\cos\theta\sin\theta + \sin^2\theta$$

Now use the identity $\cos^2\theta + \sin^2\theta = 1$ to account for the first and third terms:

$$\cos^2\theta + 2\cos\theta\sin\theta + \sin^2\theta = 1 + 2\cos\theta\sin\theta$$

as required. You have shown that $(\cos\theta + \sin\theta)^2 = 1 + 2\cos\theta\sin\theta$ and so the question is answered.

3. Deriving the important relationship $1 + \tan^2\theta = \sec^2\theta$.

Deriving this relationship gives you an excellent chance to use the other relationships and definition introduced in this guide. Let's start with $\cos^2\theta + \sin^2\theta = 1$, and divide everything by $\cos^2\theta$ to give:

$$\frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

You can simplify each term:

$\dfrac{\cos^2\theta}{\cos^2\theta}$ this is equal to 1.

$\dfrac{\sin^2\theta}{\cos^2\theta}$ this is equal to $\left(\dfrac{\sin\theta}{\cos\theta}\right)^2 = \tan^2\theta$ using the relationship derived above.

$\dfrac{1}{\cos^2\theta}$ this is equal to $\left(\dfrac{1}{\cos\theta}\right)^2 = \sec^2\theta$ using the definition on page 2.

Combining all these give the required relationship of $\boxed{1 + \tan^2\theta = \sec^2\theta}$

4. Deriving the relationship $\cot^2\theta + 1 = \csc^2\theta$.

Deriving this relationship also gives you an excellent chance to use the other relationships and definition introduced in this guide. Again let's start with $\cos^2\theta + \sin^2\theta = 1$, but this time divide everything by $\sin^2\theta$ to give:

$$\frac{\cos^2\theta}{\sin^2\theta} + \frac{\sin^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

You can simplify each term in turn:

$\dfrac{\cos^2\theta}{\sin^2\theta}$ this is equal to $\left(\dfrac{\cos\theta}{\sin\theta}\right)^2 = \cot^2\theta$ using the relationship on page 2.

$\dfrac{\sin^2\theta}{\sin^2\theta}$ this is equal to 1.

$\dfrac{1}{\sin^2\theta}$ this is equal to $\left(\dfrac{1}{\sin\theta}\right)^2 = \csc^2\theta$ using the definition on page 2.

Combining all these give the required relationship of $\boxed{\cot^2\theta + 1 = \csc^2\theta}$

Want to know more?

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