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Bridging between Algebra and Calculus

Basics of Series

This guide introduces the mathematics of series and some of the language and symbols associated with them. It describes finite and infinite series and looks at special types such as arithmetic, geometric and the harmonic series in more detail.

Introduction: sequences and series

Many fascinating and surprising results from the history of mathematics have come from the study of mathematical **series**. For example the solution to **Zeno's Paradox** about Achilles and the tortoise and Euler's famous solution to the **Basel problem** are both derived from looking at series. Series play an important role in many advanced areas of mathematics. For example **power series** are used to solve complicated differential equations, functions can be described by special series called **Taylor-Maclaurin series** and **Fourier series** offer mathematicians an alternative way of expressing wave-like functions.

There is a close association between a mathematical **sequence** (a list of numbers usually described by a pattern of some sort) and a series. You can read the study guide: *Basics of Sequences* for more details. Specifically a series is defined as the sum of the **terms** of a particular sequence.

> A series is the sum of the terms in a sequence.

Series with a pattern and the Σ notation

The majority of mathematical series follow a strict pattern or recipe, they have a formula for you to follow to generate each term. For instance, if you write the **squares of the first four natural numbers**, you get the sequence 1, 4, 9, 16. Adding these terms together results in the series:

$$1+4+9+16$$

Series which stop are called **finite series**.

Series are often long and it is impractical to write down all their terms and so a special symbol and notation is used for series which have a pattern. Mathematicians use Σ to denote the process of adding up terms. A piece of algebra follows Σ, usually in terms of the letter k (often i, j, m or n are used), which describes the pattern of the series. The letter k is called a **counter** and represents **a list of consecutive integers** which are defined by **limits** written below and above Σ. For example:

Upper limit (last value) of *k*

Pattern tells you to square *k*

$$1+4+9+16=\sum_{k=1}^{4}k^2$$

Lower limit (first value) of *k*

You could write down a series which contains the squares of *all* the natural numbers:

$1+4+9+16+\ldots$ or alternatively $\sum_{k=1}^{\infty}k^2$

The **ellipsis symbol** ('…' three dots *only*) is used to indicate that the series continues indefinitely. You should note that in the sigma notation the upper limit is infinity ∞. Series which do not have an end are called **infinite series**.

You can also use an ellipsis to fill in the gap of a long finite series. For example the series containing the squares of the positive integers from 1 to 20 can be written as:

$1+4+9+16+\ldots+400$ or alternatively $\sum_{k=1}^{20}k^2$

As you can see the sigma notation is concise and versatile, it is a good idea to be able to write series in this way.

Often you will be required to add up a series starting from the beginning. The sum of the first *n* terms of a series is commonly written as S_n. So for the series of squares:

$$S_n=\sum_{k=1}^{n}k^2 \qquad S_3=\sum_{k=1}^{3}k^2=1+4+9=14 \qquad S_{20}=\sum_{k=1}^{20}k^2=1+4+\ldots+400=2870$$

and so on.

Arithmetic series

One way of creating the terms of a series is by repeatedly adding the same number (called a **common difference**) to an initial number. For example you can make the series of the first *n* natural numbers by starting with 1 and continually adding 1 until you reach *n* to give:

$$1+2+3+4+5+\ldots+n$$

An interesting question is "what do I get when I add the first *n* counting number together?" and you can use this series to answer this question. To add this series you can use a very clever method which reveals some of the beauty of mathematics. It is said that the famous mathematician Gauss, when he was in primary school, used this method to add up the numbers 1 to 100 in under ten seconds when asked to do so by a teacher as a punishment. Begin by writing the series, including a couple of terms before the last term, and then write the series again underneath but **backwards** this time carefully lining up the terms. When you add the two series together down the columns you get:

$$\begin{array}{ccccccccccccc} 1 & + & 2 & + & 3 & +\dots+ & (n-2) & + & (n-1) & + & n & \\ n & + & (n-1) & + & (n-2) & +\dots+ & 3 & + & 2 & + & 1 & + \\ \hline (n+1) & + & (n+1) & + & (n+1) & +\dots+ & (n+1) & + & (n+1) & + & (n+1) & = 2S_n \end{array}$$

By adding in this way, you get *n* lots of $n+1$. As you added two lots of the series to get this result, the sum of the series itself is half of this. In symbols you have found the formula:

$$\boxed{S_n = \sum_{k=1}^{n} k = \frac{1}{2}n(n+1)}$$

Sum of the first *n* positive integers

Example: What is the sum of the whole numbers 1 to 100?

This is the problem that Gauss was set. Here *n* is 100 as it is the last integer and so:

$$S_{100} = 1+2+3+\dots+100 = \sum_{n=1}^{100} n = \frac{1}{2}\times 100 \times 101 = 5050$$

You can use the formula above to find the sum of **any** finite arithmetic series. Think about n arithmetic series that starts at any number, often called by *a*, and you repeatedly add a **common difference** *d*, until you reach a certain term (the n^{th} term) you would get:

$$S_n = a + (a+1d) + (a+2d) + (a+3d) + \dots + (a+(n-1)d)$$

Notice that the final term has $n-1$ lots of *d* added to *a* not *n*. This is because the first term has zero lots of *d* added. You can think of the sum of this series as adding the *a*'s together and adding the *d* 's together. As each term contributes *a* to the sum there are *n* lots of *a*. You then can factorising the remaining terms which contain *d* to get the sum:

$$S_n = na + d(1+2+3+\dots+(n-1)) = na + d\sum_{k=1}^{n-1} k$$

Here the last term on the right is the sum of the first $n-1$ positive integers, multiplied by *d*. Using the boxed formula above you get the important formula:

$$\boxed{S_n = na + \frac{1}{2}n(n-1)d}$$

Sum of the first *n* terms of an arithmetic series with first term *a* and common difference *d.*

Geometric series

You have seen that to get the next term in an arithmetic series you add a common difference to the previous term. In another common type of series, a **geometric series**, you **multiply** the previous term by a number called the **common ratio**, usually written as *r*.
For example you can start with the one and repeatedly multiply by two to generate the series:

$$1+2+4+8+\ldots=\sum_{k=1}^{\infty}2^{k-1}$$

(Remembering that anything to the power zero is one.) Just as with arithmetic series, adding up a the first *n* terms of a geometric series gives an insight into how thinking about a problem in a different way can produce an important result. Firstly write out a general, finite geometric series:

$$S_n=a+ar+ar^2+ar^3+\ldots+ar^{n-2}+ar^{n-1}$$

Now multiply both sides by *r*, this may seem an odd thing to do but it will help very shortly:

$$rS_n=ar+ar^2+ar^3+\ldots+ar^{n-1}+ar^n$$

If you subtract the second series from the first something amazing happens, all but two of the terms on the right-hand side cancel out:

$$\begin{aligned} S_n &= a+ar+ar^2+\ldots+ar^{n-2}+ar^{n-1} \\ rS_n &= ar+ar^2+ar^3+\ldots+ar^{n-1}+ar^n \quad - \\ \hline S_n-rS_n &= a-ar^n \end{aligned}$$

You can factorise both sides to get:

$$S_n(1-r)=a(1-r^n)$$

And dividing by $1-r$ gives the sum of the series S_n:

$$\boxed{S_n=\frac{a(1-r^n)}{1-r}}$$

Sum of the first *n* terms of a geometric series with first term *a* and common ratio *r*, $r\neq 1$.

The common ratio *r* cannot equal 1 as this would mean dividing by zero which is not allowed.

Example: Find $\sum_{k=1}^{10}\left(\frac{1}{2}\right)^{k-1}$.

Using the formula with $a=1$, $r=1/2$ and $n=10$ gives $S_{10}=\frac{1\times\left(1-\frac{1}{2}^{10}\right)}{1-\frac{1}{2}}=1.998046875$.

Convergence and divergence of infinite geometric series

An important property of any **infinite series** is if the result is (plus or minus) infinity or not. In the latter case you get a number when you sum the series and it is said to **converge**. If a series does not converge its sum is either plus or minus infinity and it is said to **diverge**. Working out whether a particular infinite series converges or diverges can involve advanced mathematics, however the matter is fairly straightforward for infinite geometric series:

> An infinite geometric series **converges** if $-1<r<1$,
> i.e. r is between but not equal to -1 and 1.

For all other values of r an infinite geometric series **diverges**.

> An infinite geometric series **diverges** if $r\le -1$ or $r\ge 1$,
> i.e. r is smaller than or equal to -1 or bigger than or equal to 1.

The key to understanding convergence of infinite geometric series is the value of the r^n term in the formula from the previous section. As n gets bigger (mathematically "approaches infinity") then r^n gets smaller ("approaches zero"). This is an example of a mathematical **limit**, if you want to learn more about limits you can talk to a **Learning Enhancement Tutor**. As r^n essentially becomes zero the formula for the sum in the previous section becomes:

$$S_n=\frac{a}{1-r}$$

Sum of an infinite geometric series with first term a and common ratio r, $-1<r<1$ as n approaches infinity.

Example: Does $\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^{k-1}=1+\frac{1}{2}+\frac{1}{4}+...$ converge and if so, to what value?

As $r=1/2$ you know the series converges. Also $a=1$ and so the equation above gives:

$$S_n=1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+...=\frac{1}{1-\frac{1}{2}}=2$$ as n approaches infinity.

You can use this result to resolve Zeno's Paradox.

Example: Does $\sum_{k=1}^{\infty}2^{k-1}=1+2+4+...$ converge and if so, to what value?

As the terms get bigger and bigger it should be obvious that this series diverges.

The harmonic series

It is a common misconception that all series are arithmetic or geometric. This is not the case, in fact there are many interesting series that are neither. For instance the series of unit fractions, called the **harmonic series** is very famous and much studied:

$$\sum_{k=1}^{\infty}\frac{1}{k}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots \qquad \textbf{The harmonic Series}$$

In a similar way to the geometric series $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots$, the terms in the harmonic series get smaller and smaller and you might expect that this series converges too. However this series *diverges* and the explanation is based on thinking about the series in a slightly different way. You can divide up the whole series into sections that are bigger than or equal to a half:

$$\left(1+\frac{1}{2}\right)+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\left(\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16}\right)+\ldots$$

This gives you an infinite number of halves to add together which gives infinity, in other words the harmonic series diverges.

Want to know more?

If you have any further questions about this topic you can make an appointment to see a **Learning Enhancement Tutor** in the **Student Support Service**, as well as speaking to your lecturer or adviser.

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