Learning Enhancement Team

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STUDENT SUPPORT SERVICE

Model answers: Basics of Series

Basics of Series study guide

1. a) $S = \sum_{k=1}^{5} k^3$

The question is asking for the series of the first five cube numbers to be written in sigma notation. You can see here that the lower limit of k is 1, the upper limit of k is 5, and that the series is of cube numbers, where the k^{th} cube number is written k^3.

b) $S = \sum_{k=1}^{20} \frac{1}{2}k(k+1)$

The question is asking for the series of the first twenty triangle numbers to be written in sigma notation. You can see here that the lower limit of k is 1, the upper limit of k is 20, and that the series is of triangle numbers, where the k^{th} triangle number is given by the formula $\frac{1}{2}k(k+1)$.

c) $S = \sum_{k=1}^{n} 3^k$

The question is asking for the series of the first n powers of 3 to be written in sigma notation. You can see here that the lower limit of k is 1. The upper limit of k is n in this case, so you can write n where the upper limit goes. The series is of powers of 3, and the k^{th} power of 3 is written 3^k.

d) $S = \sum_{k=1}^{\infty} \sqrt{k}$

The question is asking for the infinite series $1 + \sqrt{2} + \sqrt{3} + 2 + \sqrt{5} + \ldots$ to be written in sigma notation. You can see that this is just the sum of all square roots. Using k, you can see that the lower limit of k is 1. As this series is infinite, there is no upper limit of k, so you can use ∞ in place of the upper limit to show that the series is infinite. The series is of square roots of k, and the k^{th} square root is written $\sqrt{k}$.

2. The following formula is used in this question:

$$S_n = na + \frac{1}{2}n(n-1)d$$

where a is the starting number, d is the common difference and n is the number of terms we are adding.

a) For the series $2+8+14+20+26+\ldots$, $S_{11} = 352$, $S_{21} = 802$ and $S_{41} = 5002$.

You can see that this series has starting number 2 and a common difference of 6. To work out S_{11} you can use the formula with $a = 2$, $d = 6$, and $n = 11$ to get

$$S_{11} = 11 \cdot 2 + \frac{1}{2} \cdot 11 \cdot (10) \cdot 6$$

$$S_{11} = 22 + 330$$

$$S_{11} = 352$$

For S_{21} you can use the formula with $a = 2$, $d = 6$, and $n = 21$ to get

$$S_{21} = 21 \cdot 2 + \frac{1}{2} \cdot 21 \cdot (20) \cdot 6$$

$$S_{21} = 42 + 760$$

$$S_{21} = 802$$

For S_{41} you can use the formula with $a = 2$, $d = 6$, and $n = 41$ to get

$$S_{41} = 41 \cdot 2 + \frac{1}{2} \cdot 41 \cdot (40) \cdot 6$$

$$S_{41} = 82 + 4920$$

$$S_{41} = 5002$$

b) For the series $100+210+320+430+540+\ldots$, $S_{11} = 6750$, $S_{21} = 25200$ and $S_{41} = 94300$.

You can see that this series has starting number 100 and a common difference of 110. To work out S_{11} you can use the formula with $a = 100$, $d = 110$, and $n = 11$ to get

$$S_{11} = 11 \cdot 100 + \frac{1}{2} \cdot 11 \cdot (10) \cdot 110$$

$$S_{11} = 1100 + 5650$$

$$S_{11} = 6750$$

For S_{21} you can use the formula with $a = 100$, $d = 110$, and $n = 21$ to get

$$S_{21} = 21 \cdot 100 + \frac{1}{2} \cdot 21 \cdot (20) \cdot 110$$

$$S_{21} = 2100 + 23100$$

$$S_{21} = 25200$$

For S_{41} you can use the formula with $a = 100$, $d = 110$, and $n = 41$ to get

$$S_{41} = 41 \cdot 100 + \frac{1}{2} \cdot 41 \cdot (40) \cdot 110$$

$$S_{41} = 4100 + 90200$$

$$S_{41} = 94300.$$

c) For the series $300 + 285 + 270 + 255 + 240 + \ldots$, $S_{11} = 2475$, $S_{21} = 3150$ and $S_{41} = 0$.

You can see that this series has starting number 300 and a common difference of -15. To work out S_{11} you can use the formula with $a = 300$, $d = -15$, and $n = 11$ to get

$$S_{11} = 11 \cdot 300 + \frac{1}{2} \cdot 11 \cdot (10) \cdot (-15)$$

$$S_{11} = 3300 + (-825)$$

$$S_{11} = 2475$$

For S_{21} you can use the formula with $a = 300$, $d = -15$, and $n = 21$ to get

$$S_{21} = 21 \cdot 300 + \frac{1}{2} \cdot 21 \cdot (20) \cdot (-15)$$

$$S_{21} = 6300 + (-3150)$$

$$S_{21} = 3150$$

For S_{41} you can use the formula with $a = 300$, $d = -15$, and $n = 41$ to get

$$S_{41} = 41 \cdot 300 + \frac{1}{2} \cdot 41 \cdot (40) \cdot (-15)$$

$$S_{41} = 12300 + (-12300)$$

$$S_{41} = 0$$

d) For the series $1 + \frac{3}{2} + 2 + \frac{5}{2} + 3 + \ldots$, $S_{11} = 38.5$, $S_{21} = 126$ and $S_{41} = 451$.

You can see that this series has starting number 1 and a common difference of $1/2$. To work out S_{11} you can use the formula with $a = 1$, $d = 1/2$, and $n = 11$ to get

$$S_{11} = 11 \cdot 1 + \frac{1}{2} \cdot 11 \cdot (10) \cdot \left(\frac{1}{2}\right)$$

$$S_{11} = 11 + 27.5$$

$$S_{11} = 38.5$$

For S_{21} you can use the formula with $a = 1$, $d = \frac{1}{2}$, and $n = 21$ to get

$$S_{21} = 21 \cdot 1 + \frac{1}{2} \cdot 21 \cdot (20) \cdot \left(\frac{1}{2}\right)$$

$$S_{21} = 21 + 105$$

$$S_{21} = 126$$

For S_{41} you can use the formula with $a = 1$, $d = \frac{1}{2}$, and $n = 41$ to get

$$S_{41} = 41 \cdot 1 + \frac{1}{2} \cdot 41 \cdot (40) \cdot \left(\frac{1}{2}\right)$$

$$S_{41} = 41 + 410$$

$$S_{41} = 451$$

3. The following formula is used in this question:

$$S_n = \frac{a(1 - r^n)}{(1 - r)}$$

where a is the starting number, r is the common ratio and n is the number of terms we are adding. Remember you should give your answer to three decimal places, rounding only when the calculation is complete.

a) For the series $1+\frac{3}{2}+\frac{9}{4}+\frac{27}{8}+\frac{81}{16}+\ldots$, $S_5 = 13.188$, $S_{11} = 170.995$ and $S_{21} = 9973.770$.

You can see that this series has starting number 1 and a common ratio of $3/2$. To work out S_5 you can use the formula with $a = 1$, $r = 3/2$, and $n = 5$ to get

$$S_5 = \frac{1\cdot\left(1-\left(\frac{3}{2}\right)^5\right)}{1-\frac{3}{2}}$$

$$S_5 = \frac{1\cdot(1-7.594)}{1-\frac{3}{2}}$$

$$S_5 = \frac{-6.594}{-0.5}$$

$$S_5 = 13.188$$

For S_{11} you can use the formula with $a = 1$, $r = 3/2$, and $n = 11$ to get

$$S_{11} = \frac{1\cdot\left(1-\left(\frac{3}{2}\right)^{11}\right)}{1-\frac{3}{2}}$$

$$S_{11} = \frac{1\cdot(1-86.498)}{1-\frac{3}{2}}$$

$$S_{11} = \frac{-85.498}{-0.5}$$

$$S_{11} = 170.995$$

For S_{21} you can use the formula with $a = 1$, $r = 3/2$, and $n = 21$ to get

$$S_{21} = \frac{1 \cdot \left(1 - \left(\frac{3}{2}\right)^{21}\right)}{1 - \frac{3}{2}}$$

$$S_{21} = \frac{1 \cdot (1 - 4987.885)}{1 - \frac{3}{2}}$$

$$S_{21} = \frac{-4986.885}{-0.5}$$

$$S_{21} = 9973.770$$

b) For the series $\sum_{k=1}^{\infty} \left(\frac{5}{6}\right)^{k-1}$, $S_5 = 3.589$, $S_{11} = 5.192$ and $S_{21} = 5.870$.

You can see that this series has starting number 1 (as anything to the power of 0 is 1) and a common ratio of $5/6$. To work out S_5 you can use the formula with $a = 1$, $r = 5/6$, and $n = 5$ to get

$$S_5 = \frac{1 \cdot \left(1 - \left(\frac{5}{6}\right)^{5}\right)}{1 - \frac{5}{6}}$$

$$S_5 = \frac{1 \cdot (1 - 0.402)}{1 - \frac{5}{6}}$$

$$S_5 = \frac{0.598}{0.183}$$

$$S_5 = 3.589$$

For S_{11} you can use the formula with $a = 1$, $r = 5/6$, and $n = 11$ to get

$$S_{11} = \frac{1 \cdot \left(1 - \left(\frac{5}{6}\right)^{11}\right)}{1 - \frac{5}{6}}$$

$$S_{11} = \frac{1 \cdot (1 - 0.135)}{1 - \frac{5}{6}}$$

$$S_{11} = \frac{0.865}{0.183}$$

$$S_{11} = 5.192$$

For S_{21} you can use the formula with $a = 1$, $r = 5/6$, and $n = 21$ to get

$$S_{21} = \frac{1 \cdot \left(1 - \left(\frac{5}{6}\right)^{21}\right)}{1 - \frac{5}{6}}$$

$$S_{21} = \frac{1 \cdot (1 - 0.022)}{1 - \frac{5}{6}}$$

$$S_{21} = \frac{0.978}{0.183}$$

$$S_{21} = 5.870$$

c) For the series $1 + 4 + 16 + 64 + 256 + \ldots$, $S_5 = 341$, $S_{11} = 1398101$ and $S_{21} = 1.466 \times 10^{12}$.

You can see that this series has starting number 1 and a common ratio of 4. To work out S_5 you can use the formula with $a = 1$, $r = 4$, and $n = 5$ to get

$$S_5 = \frac{1 \cdot (1 - 4^5)}{1 - 4}$$

$$S_5 = \frac{1 \cdot (1 - 1024)}{1 - 4}$$

$$S_5 = \frac{-1023}{-3}$$

$$S_5 = 341$$

For S_{11} you can use the formula with $a=1$, $r=4$, and $n=11$ to get

$$S_{11}=\frac{1\cdot\left(1-4^{11}\right)}{1-4}$$

$$S_{11}=\frac{1\cdot\left(1-4194304\right)}{1-4}$$

$$S_{11}=\frac{-4194303}{-3}$$

$$S_{11}=1398101$$

For S_{21} you can use the formula with $a=1$, $r=4$, and $n=21$ to get

$$S_{21}=\frac{1\cdot\left(1-4^{21}\right)}{1-4}$$

$$S_{21}=\frac{1\cdot\left(1-4.398\times10^{12}\right)}{1-4}$$

$$S_{21}=\frac{-4.398\times10^{12}}{-3}$$

$$S_{21}=1.466\times10^{12}$$

d) For the series $243-81+27-9+3-\ldots$, $S_5=183$, $S_{11}=182.257$ and $S_{21}=182.250$.

You can see that this series has starting number 243 and a common ratio of $-1/3$. To work out S_5 you can use the formula with $a=243$, $r=-1/3$, and $n=5$ to get

$$S_5=\frac{243\cdot\left(1-\left(-\frac{1}{3}\right)^5\right)}{1-\left(-\frac{1}{3}\right)}$$

$$S_5=\frac{243\cdot\left(1-\left(-0.004\right)\right)}{1-\left(-\frac{1}{3}\right)}$$

$$S_5=\frac{244}{1.333}$$

$$S_5=183$$

For S_{11} you can use the formula with $a = 243$, $r = -1/3$, and $n = 11$ to get

$$S_{11} = \frac{243 \cdot \left(1 - \left(-\frac{1}{3}\right)^{11}\right)}{1 - \left(-\frac{1}{3}\right)}$$

$$S_{11} = \frac{243 \cdot \left(1 - \left(-5.645 \times 10^{-6}\right)\right)}{1 - \left(-\frac{1}{3}\right)}$$

$$S_{11} = \frac{243.001}{1.333}$$

$$S_{11} = 182.251$$

For S_{21} you can use the formula with $a = 243$, $r = -1/3$, and $n = 21$ to get

$$S_{21} = \frac{243 \cdot \left(1 - \left(-\frac{1}{3}\right)^{21}\right)}{1 - \left(-\frac{1}{3}\right)}$$

$$S_{21} = \frac{243 \cdot \left(1 - \left(-9.560 \times 10^{-11}\right)\right)}{1 - \left(-\frac{1}{3}\right)}$$

$$S_{21} = \frac{243.000}{1.333}$$

$$S_{21} = 182.250.$$

e) For the series $\sum_{k=1}^{\infty} 9\left(\frac{1}{10}\right)^{k-1}$, $S_5 = 9.999$, $S_{11} = 9.999$ and $S_{21} = 9.999$.

You can see that this series has starting number 9 (as anything to the power of 0 is 1) and a common ratio of $1/10$. To work out S_5 you can use the formula with $a = 9$, $r = 1/10$, and $n = 5$ to get

$$S_5 = \frac{9 \cdot \left(1 - \left(\frac{1}{10}\right)^5\right)}{1 - \frac{1}{10}}$$

$$S_5 = \frac{9 \cdot \left(1 - 10^{-5}\right)}{1 - \frac{1}{10}}$$

$$S_5 = \frac{8.99991}{0.9}$$

$$S_5 = 9.99999 = 9.999 \quad (3 \text{ d.p.}).$$

For S_{11} you can use the formula with $a = 9$, $r = 1/10$, and $n = 11$ to get

$$S_{11} = \frac{9 \cdot \left(1 - \left(\frac{1}{10}\right)^{11}\right)}{1 - \frac{1}{10}}$$

$$S_{11} = \frac{1 \cdot \left(1 - 10^{-11}\right)}{1 - \frac{1}{10}}$$

$$S_{11} = \frac{8.9999999999\,1}{0.9}$$

$$S_{11} = 9.9999999999\,9 = 9.999 \quad (3 \text{d.p.}).$$

For S_{21} you can use the formula with $a = 9$, $r = 1/10$, and $n = 21$ to get

$$S_{21} = \frac{9 \cdot \left(1 - \left(\frac{1}{10}\right)^{21}\right)}{1 - \frac{1}{10}}$$

$$S_{21} = \frac{9 \cdot \left(1 - 10^{-21}\right)}{1 - \frac{1}{10}}$$

$$S_{21} = \frac{8.9999999999\,9999999999\,1}{0.9}$$

$$S_{21} = 9.9999999999\,9999999999\,9 = 9.999 \quad (3 \text{d.p.}).$$

Remember that a geometric series with common ratio r is convergent if $-1 < r < 1$. A geometric series with common ratio r is divergent if $r \leq -1$ or $r \geq 1$. You can look at each series in turn to decide whether they are convergent or divergent.

You have already noticed that the geometric series $1+\frac{3}{2}+\frac{9}{4}+\frac{27}{8}+\frac{81}{16}+\ldots$ has common ratio $r=3/2$. As $(3/2)\geq 1$, you can see that this series is divergent.

You have already noticed that the geometric series $\sum_{k=1}^{\infty}\left(\frac{5}{6}\right)^{k-1}$ has common ratio $r=5/6$. As $-1<(5/6)<1$, you can see that this series is convergent.

You have already noticed that the geometric series $1+4+16+64+256+\ldots$ has common ratio $r=4$. As $4\geq 1$, you can see that this series is divergent.

You have already noticed that the geometric series $243-81+27-9+3-\ldots$ has common ratio $r=-1/3$. As $-1<(-1/3)<1$, you can see that this series is convergent.

You have already noticed that the geometric series $\sum_{k=1}^{\infty}9\left(\frac{1}{10}\right)^{k-1}$ has common ratio $r=1/10$. As $-1<(1/10)<1$, you can see that this series is convergent.

4. The following formula is used in this question:

$$S_n=\frac{a}{1-r}$$

where *a* is the starting number and *r* is the common ratio. This formula is used for convergent geometric series, and is the value of that series as *n* approaches infinity. You have already decided which of the geometric series in question 3 are convergent.

For the convergent series $\sum_{k=1}^{\infty}\left(\frac{5}{6}\right)^{k-1}$, you can use the formula with $a=1$ and $r=5/6$ to get:

$$S_n=\frac{1}{1-\frac{5}{6}}=\frac{1}{\frac{1}{6}}=6 \qquad \text{as } n \text{ approaches infinity.}$$

For the convergent series $243-81+27-9+3-\ldots$, you can use the formula with $a=243$ and $r=-1/3$ to get:

$$S_n = \frac{243}{1-\left(-\frac{1}{3}\right)} = \frac{243}{\frac{4}{3}} = 182.25 \qquad \text{as } n \text{ approaches infinity}$$

For the convergent series $\sum_{k=1}^{\infty} 9\left(\frac{1}{10}\right)^k$, you can use the formula with $a = 9$ and $r = 1/10$ to get:

$$S_n = \frac{9}{1-\frac{1}{10}} = \frac{9}{0.9} = 10 \qquad \text{as } n \text{ approaches infinity.}$$

and so you have proved that $9.999\ldots = 10$.